JOEL LEWIS: Hi. Welcome back to recitation.

Today I wanted to talk about something that's mentioned in the notes but wasn't covered in lecture because of the exam review. So this is, the subject is hyperbolic trig functions. So first I just wanted to define them for you and graph them so we can get a little bit of a feeling for what these functions are like, and then I'm going to explain to you why they have the words hyperbolic and trig in their names. So these are some interesting functions. They're not, they don't-- aren't quite as important as your usual, sort of circular trig functions. But yeah, so let me introduce them and let me jump in just with their definition.

So there are two most important ones. Just like a regular trigonometric functions there's the sine and the cosine and then you can write the other four trigonometric functions in terms of them. So for hyperbolic trig functions we have the hyperbolic cosine and the hyperbolic sine. So the notation here, we write c o s h. So the h for hyperbolic. So hyperbolic cosine. And usually we pronounce this "cosh." And similarly, for the hyperbolic sine we write s i n h, for hyperbolic sine, except in the reverse order. And we usually pronounce this "sinch," so in American English as if there were an extra c in there. Sinch.

OK. So these functions have fairly simple definitions in terms of the exponential function, e to the x. So cosh of x is defined to be e to the x plus e to the minus x divided by 2. And sinh of x is defined to be e to x minus e to the minus x divided by 2. So if you remember what your graph of e to the x looks like, and your graph of e to the minus x, it's not hard to see that the graphs of cosh x and sinh x should look sort of like this.

So for cosh x, so we see as x gets big, so e to the minus x is going to 0. It's not very important. So it mostly is driven by this e to the x part. And as x gets negative and big, then this is going to 0 and this is getting larger and larger. So we got something that looks like this. So it looks a little bit, in this picture it looks a little bit like a parabola, but the growth here is exponential at both sides. So in fact, this is growing much, much, much faster than, say, 1 plus x squared. So it's a much steeper curve. OK. And it reaches its minimum here at x equals 0-- it has the value 1 plus 1 over 2. So its minimum there is at x equals 0, it has its minimum value 1.

For sinh, OK, so we're taking the difference of them. So it's similar when x is positive and large, e to the x is big, and e to the minus x is pretty small, almost negligible. So we got exponential growth off that side. When x becomes negative and large, e to the x is going to 0, e to the minus x is becoming large, but it's-- we've got a minus sign here. So as x goes to minus infinity, this curve goes also to minus infinity. And again, the growth here is exponential in both cases. And if you were curious, say about what the slope there at the origin is, you could quickly take a derivative and check that that's passing through the origin with slope 1 there.

OK. So this is a sort of basic picture of what these curves look like.

They have some nice properties, and let me talk about them. So for example, one nice thing you might notice about these functions is that it's easy to compute their derivatives. Right? So if we look at d/dx of cosh x, in order to compute that, well, just look at the definition of cosh. So it's really just a sum of two exponential functions. Exponential functions are easy to take the derivatives. Take the derivative of e to the x, you get e to the x. Take the derivative of e to the minus x, well, OK, so it's a little chain rule, so you get a minus 1 in front. So the derivative of cosh x is e to the x minus e to the minus x over 2.

But we have a name for this. This is actually just sinh x. So the derivative of cosh is sinh, and the derivative of sinh, well, OK. You look at the same thing, take this formula, take its derivative. Well, e to the x, take its derivative, you get e to the x. e to the minus x, take its derivative, you get minus e to the minus x, so those two minus signs cancel out and become a plus. So this is e to the x plus e to the minus x over 2, which is cosh x.

So here you have some behavior that's a little bit reminiscent of the behavior of trig functions. Right? For trig functions, if you take the derivative of sine you get cosine. And if you take the derivative of cosine you almost get back sine, but you get minus sine. So here you don't have that extra negative sign floating around. Right? So you, when you take the derivative of cosh you get sinh on the nose. No minus sign needed.

So that's interesting. But the real reason that these have the words trig in their name is actually a little bit deeper. So let me come over here and draw a couple pictures. So the normal trig functions-- what sometimes we call the circular trig functions if we want to distinguish them from the hyperbolic trig functions-- they're closely-- so circular trig functions, they're closely related to the unit circle. So the unit circle has equation x squared plus y squared equals 1. It's a circle. Well, close enough, right? And what is the nice relationship between this circle and the trig functions? Well, if you choose any point on this circle, then there exists some value of t such that this point has coordinates cosine t comma sine t.

Now it happens that the value of t is actually the angle that that radius makes with the positive axis. But not going to worry about that right now. It's not the key idea of import. So as t varies through the real numbers, the point cosine t, sine t, that varies and it just goes around this curve. So it traces out this circle exactly.

So the hyperbolic trig functions show up in a very similar situation. But instead of looking at the unit circle, what we want to look at is the unit rectangular hyperbola. So what do I mean by that? Well, so instead of taking the equation x squared plus y squared equals 1, which gives a circle, I'm going to look at a very similar equation that gives a hyperbola. So this is the equation x squared minus y squared equals 1.

So if you if you graph this equation, what you'll see is that, well, it passes through the point (1, 0). And then we've got one branch here, we've got a little asymptote there. So it's got a right branch like that, and also it's symmetric across the y-axis. So there's a symmetric left branch here. So this is the graph of the equation x squared minus y squared equals 1. So it's this hyperbola.

Now what I claim is that cosh and sinh have the same relationship to this hyperbola as cosine and sine have to the circle. Well, so I'm fudging a little bit. So it turns out it's only the right half of the hyperbola. So what do I mean by that?

Well, here's what I'd like to do. Set x equals-- so we're going to introduce a new variable, u-- I'm going to set x equal cosh u and y equals sinh u. And I'm going to look at the quantity x squared minus y squared. So x squared minus y squared. So this is, so we use most of the same notations for hyperbolic trig functions that we do for regular trig functions. So this is cosh squared u minus sinh squared u.

And now we can plug in the formulas for cosh and sinh that we have. So this is equal to e to the u plus e to the minus u over 2, quantity squared, minus e to the u minus e to the minus u over 2, quantity squared. And now we can expand out both of these factors and-- both of these squares, rather, and put them together. So over 2 squared is over 4 and we square this and we get e to the 2u. OK, so then we get 2 times e to the u times e to the minus u. But e to the u times e to the minus u is just 1, so plus 2. Plus e to the minus 2u minus e to the 2u minus 2 plus e to the minus 2u-- so same thing over here-- over 4. OK, so the e to the 2u's cancel and the e to the minus 2u's cancel and we're left with 2 minus minus 2. That's 4. So this is 4 over 4, so this is equal to 1.

OK. So if x is equal to cosh u and y is equal to sinh u, then x squared minus y squared is equal to 1. So if we choose a point (cosh u, sinh u) for some u, that point lies on this hyperbola. That's what this says. That this point-- OK, so the point (cosh u, sinh u) is somewhere on this hyperbola.

And what's also true is the sort of reverse statement. If you look at all such points, if you let u vary and look-- through the real numbers and you ask what happens to this point (cosh u, sinh u), the answers is that it traces out the right half of this hyperbola. If you go back to the graph of y equals cosh x, you'll see that the hyperbolic cosine function is always positive. So we can't-- over here, we can't trace out this left branch where x is negative. Although it's easy enough to say what does trace out this left branch. Since it's just the mirror image, this is traced out by minus cosh u comma sinh u.

So there's a-- so the hyperbolic trig functions have the same relationship to this branch of this hyperbola that the regular trig functions have to the circle. So there's where the words hyperbolic and trig functions come from.

So let me say one more thing about them, which is that we saw that they have this analogy with regular trig functions. Right? So instead of satisfying cosine squared plus sine squared equals 1, they satisfy cosh squared minus sinh squared equals 1. And instead of satisfying the derivative of sine equals cosine and the derivative of cosine equals minus sine, they satisfy derivative of cosh equals sinh and derivative of sinh equals cosh. So similar relationships. Not exactly the same, but similar. So this is true of a lot of trig relationships, that there's a corresponding formula for the hyperbolic trig functions.

So one example of such a formula is your-- for example, your angle addition formulas. So I'm going to just leave this is an exercise for you. So let me, I guess I'll just stick it in this funny little piece of board right here.

So, exercise. Find sinh of x plus y and cosh of x plus y in terms of sinh x, cosh x, sinh y, and cosh y. So in other words, find the corresponding formula to the angle addition formula in that case of the hyperbolic trig functions.

So I'll leave you with that.