

## Lemniscate

The curve described in polar coordinates by  $r^2 = \cos(2\theta)$  is called a *lemniscate*.

- For what values of  $\theta$  does there exist such a point  $(r, \theta)$ ?
- For what values of  $\theta$  is  $r$  at its minimum length?
- For what values of  $\theta$  is  $r$  at its maximum length?
- Use the information you have gathered to sketch a rough graph of this curve.

## Solution

It is helpful to have a graph of  $\cos(2\theta)$  to look at while working this problem.

- For what values of  $\theta$  does there exist such a point  $(r, \theta)$ ?

We can't take the square root of a negative number (in this class), so  $r$  is undefined anywhere  $\cos(2\theta)$  is negative. The function  $f(\theta) = \cos(\theta)$  has negative outputs on the interval  $\pi/2 < \theta < 3\pi/2$  and in general on all intervals of width  $\pi$  centered on an odd multiple of  $\pi$ . The graph of the function  $g(\theta) = \cos(2\theta)$  is a horizontally compressed copy of the graph of  $f(\theta)$ , so  $\cos(2\theta)$  is zero on the intervals of width  $\pi/2$  centered on odd multiples of  $\pi/2$ ; on the intervals  $\pi/4 < \theta < 3\pi/4$  and  $5\pi/4 < \theta < 7\pi/4$ , etc.

For all other values of  $\theta$ , the points  $(\sqrt{\cos(2\theta)}, \theta)$  and  $(-\sqrt{\cos(2\theta)}, \theta)$  are points on the lemniscate.

- For what values of  $\theta$  is  $r$  at its minimum length?

When  $\cos(2\theta) = 0$ ,  $r(\theta) = 0$ . This is the smallest value  $r$  can attain (because  $r$  is defined in terms of a square root function). Considering the graph  $\cos(2\theta)$ , we observe that  $\cos(2\theta) = 0$  when  $2\theta$  is an odd multiple of  $\pi/2$ ; i.e. when  $\theta$  is an odd multiple of  $\pi/4$ .

The radius  $r$  has length 0 when  $\theta = \dots, -3\pi/4, -\pi/4, \pi/4, 3\pi/4, \dots$

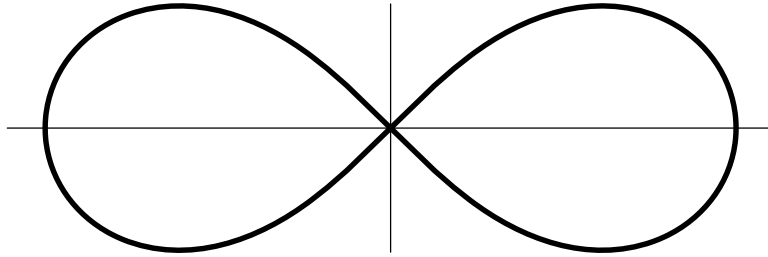
- For what values of  $\theta$  is  $r$  at its maximum length?

When  $\cos(2\theta) = 1$ ,  $r = \pm 1$ . The value of  $\cos(2\theta)$  is never greater than 1, so  $r = \pm\sqrt{\cos(2\theta)}$  is never longer than 1 unit.

Going back to our graph of  $\cos(2\theta)$ , we see that the radius  $r$  is greatest when  $\theta$  is a multiple of  $\pi$ :  $\theta = \dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots$

d) Use the information you have gathered to sketch a rough graph of this curve.

We know that the curve  $(r, \theta)$  sweeps out a path for the intervals  $k\pi - \pi/4 < \theta < k\pi + \pi/4$  and does not exist outside of those intervals. On each of these intervals,  $r$  starts at its minimum value of 0, increases to its maximum 1, then decreases again. We could try to find coordinates for the points  $(r, \theta)$  when  $\theta = k\pi \pm \pi/8$ , or we could make a guess at the path the curve follows between extremes. Our end result should look a little bit like the figure below.



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