Lemniscate

The curve described in polar coordinates by $r^2 = \cos(2\theta)$ is called a *lemniscate*.

a) For what values of θ does there exist such a point (r, θ)?

b) For what values of θ is r at its minimum length?

c) For what values of θ is r at its maximum length?

d) Use the information you have gathered to sketch a rough graph of this curve.

Solution

It is helpful to have a graph of $\cos(2\theta)$ to look at while working this problem.

a) For what values of θ does there exist such a point (r, θ)?

We can't take the square root of a negative number (in this class), so r is undefined anywhere $\cos(2\theta)$ is negative. The function $f(\theta) = \cos(\theta)$ has negative outputs on the interval $\pi/2 < \theta < 3\pi/2$ and in general on all intervals of width π centered on an odd multiple of π. The graph of the function $g(\theta) = \cos(2\theta)$ is a horizontally compressed copy of the graph of $f(\theta)$, so $\cos(2\theta)$ is zero on the intervals of width $\pi/2$ centered on odd multiples of $\pi/2$; on the intervals $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$, etc.

For all other values of θ, the points $\left(\sqrt{\cos(2\theta)}, \theta\right)$ and $\left(-\sqrt{\cos(2\theta)}, \theta\right)$ are points on the lemniscate.

b) For what values of θ is r at its minimum length?

When $\cos(2\theta) = 0$, $r(\theta) = 0$. This is the smallest value r can attain (because r is defined in terms of a square root function). Considering the graph $\cos(2\theta)$, we observe that $\cos(2\theta) = 0$ when 2θ is an odd multiple of $\pi/2$; i.e. when θ is an odd multiple of $\pi/4$.

The radius r has length 0 when $\theta = ..., -3\pi/4, -\pi/4, \pi/4, 3\pi/4, ...$

c) For what values of θ is r at its maximum length?

When $\cos(2\theta) = 1$, $r = \pm 1$. The value of $\cos(2\theta)$ is never greater than 1, so $r = \pm\sqrt{\cos(2\theta)}$ is never longer than 1 unit.

Going back to our graph of $\cos(2\theta)$, we see that the radius r is greatest when θ is a multiple of π: $\theta = ..., -2\pi, -\pi, 0, \pi, 2\pi, ...$

d) Use the information you have gathered to sketch a rough graph of this curve.

We know that the curve (r, θ) sweeps out a path for the intervals $k\pi - \pi/4 < \theta < k\pi + \pi/4$ and does not exist outside of those intervals. On each of these intervals, r starts at its minimum value of 0, increases to its maximum 1, then decreases again. We could try to find coordinates for the points (r, θ) when $\theta = k\pi \pm \pi/8$, or we could make a guess at the path the curve follows between extremes. Our end result should look a little bit like the figure below.