A Formula for e

We calculated that if $a_k = (1 + \frac{1}{k})^k$, then $\lim_{k\to\infty} a_k = e$ by first showing that $\lim_{k\to\infty} \ln a_k = 1$. Since $e^{\ln a_k} = a_k$, as k goes to infinity $a_k = e^{\ln a_k}$ will tend toward $e^1 = e$.

The key point here was that $a_k = e^{\ln a_k}$; that the natural log function is the inverse of the exponential function.

Question: Shouldn't $\ln a_k$ tend towards zero, because a_k tends toward 1?

Answer: It's true that $(1+\frac{1}{k})$ tends toward 1, and so $\ln(1+\frac{1}{k})$ tends toward 0. But that's not the limit we want; we're asking about $\ln a_k = k \cdot \ln(1 + \frac{1}{k})$. As $\ln(1 + \frac{1}{k})$ is tending toward 0, k tends toward infinity. That's why we needed to use limits and derivatives to figure out what the limit of this expression was.

We know that $\lim_{k\to\infty} a_k = e$, and all equalities can be read in both directions. So $e = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k$. In other words, this limit is a formula for e. By looking at the formula from a different angle, we discover that we can use the expression $\left(1 + \frac{1}{k}\right)^k$ to compute a base e for which graph of e^x has slope 1 when $x = 0$.

Often we can improve our understanding of mathematics by looking at things in several different ways, and that's what we're going to be doing at the end of this lecture on derivatives.