Limits at Infinity of $\dfrac{e^x}{x}$ and $\dfrac{x}{e^x}$

As x approaches infinity, the rational expressions $\dfrac{e^x}{x}$ and $\dfrac{x}{e^x}$ take on the form $\frac{\infty}{\infty}$. Use the extended version of l'Hopital's rule to evaluate the following limits, if they exist.

a) $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x}$

b) $\displaystyle\lim_{x\to+\infty} \frac{x}{e^x}$

Solution

The extended version of l'Hopital's rule tells us that:

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$$

provided that $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ have the appropriate properties and $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$ exists. Since we have verified that the limits in question are of the form $\frac{\infty}{\infty}$, we may apply this rule.

a) $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x}$

Because this is of the form $\frac{\infty}{\infty}$, we know that if the limit exists then:

$$\lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} \frac{e^x}{1} = \infty.$$

As x approaches infinity, $\frac{e^x}{1}$ does as well; hence so does $\frac{e^x}{x}$.

b) $\displaystyle\lim_{x\to+\infty} \frac{x}{e^x}$

If the limit exists then:

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0.$$

As x approaches infinity, the denominator of this rational function grows faster than the numerator and so the limit of the quotient is 0.

We will soon see how limits such as these can be used to compare the *rates of growth* of two functions.