Taylor's Series of $\sin x$

In order to use Taylor's formula to find the power series expansion of $\sin x$ we have to compute the derivatives of $\sin(x)$:

$$\begin{aligned}
\sin'(x) &= \cos(x) \\
\sin''(x) &= -\sin(x) \\
\sin'''(x) &= -\cos(x) \\
\sin^{(4)}(x) &= \sin(x).
\end{aligned}$$

Since $\sin^{(4)}(x) = \sin(x)$, this pattern will repeat.

Next we need to evaluate the function and its derivatives at 0:

$$\begin{aligned}
\sin(0) &= 0 \\
\sin'(0) &= 1 \\
\sin''(0) &= 0 \\
\sin'''(0) &= -1 \\
\sin^{(4)}(0) &= 0.
\end{aligned}$$

Again, the pattern repeats.

Taylor's formula now tells us that:

$$\begin{aligned}
\sin(x) &= 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}$$

Notice that the signs alternate and the denominators get very big; factorials grow very fast.

The radius of convergence R is infinity; let's see why. The terms in this sum look like:

$$\frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{(2n+1)}.$$

Suppose x is some fixed number. Then as n goes to infinity, the terms on the right in the product above will be very, very small numbers and there will be more and more of them as n increases.

In other words, the terms in the series will get smaller as n gets bigger; that's an indication that x may be inside the radius of convergence. But this would be true for any fixed value of x, so the radius of convergence is infinity.

Why do we care what the power series expansion of $\sin(x)$ is? If we use enough terms of the series we can get a good estimate of the value of $\sin(x)$ for any value of x.

This is very useful information about the function $\sin(x)$ but it doesn't tell the whole story. For example, it's hard to tell from the formula that $\sin(x)$ is periodic. The period of $\sin(x)$ is 2π; how is this series related to the number π?

Power series are very good for some things but can also hide some properties of functions.