Integrating $\dfrac{1}{(5x+2)^2}$ from 1 to infinity

Compute: $\displaystyle\int_1^\infty \frac{dx}{(5x+2)^2}$

Solution

This is an improper integral; our first step is to convert it to a proper one:

$$\int_1^\infty \frac{dx}{(5x+2)^2} = \lim_{N\to\infty} \int_1^N \frac{dx}{(5x+2)^2}$$

We can now apply a substitution to evaluate the integral. If we let $u = 5x+2$ then $du = 5\,dx$ and $dx = \frac{1}{5}\,du$. When $x = 1$, $u = 7$ and when $x = N$, $u = 5N+2$. Therefore:

$$\begin{aligned}
\int_1^N \frac{dx}{(5x+2)^2} &= \int_7^{2N+5} \frac{\frac{1}{5}\,du}{u^2} \\
&= \frac{1}{5}\int_7^{5N+2} u^{-2}\,du \\
&= \frac{1}{5}(-u^{-1})\Big|_7^{5N+2} \\
&= \frac{1}{5}\left(-\frac{1}{5N+2} - \left(-\frac{1}{7}\right)\right) \\
&= \frac{1}{35} - \frac{1}{5(5N+2)}
\end{aligned}$$

By evaluating the limit, we find the value of the improper integral:

$$\begin{aligned}
\int_1^\infty \frac{dx}{(5x+2)^2} &= \lim_{N\to\infty} \int_1^N \frac{dx}{(5x+2)^2} \\
&= \lim_{N\to\infty} \left(\frac{1}{35} - \frac{1}{5(5N+2)}\right) \\
&= \frac{1}{35} - 0 \\
&= \frac{1}{35} \approx .03
\end{aligned}$$

Geometrically, this tells us that if we horizontally compress the graph of $\frac{1}{x^2}$ (by multiplying x by 5) and then shift the result to the left 2 units, the final graph is very close to the x-axis for $x > 1$.