# 18.014 Problem Set 7 Solutions

Total: 12 points

**Problem 1:** Given a function $g$ continuous everywhere such that $g(1) = 5$ and $\int_0^1 g(t)dt = 2$, let $f(x) = \frac{1}{2}\int_0^x (x-t)^2 g(t)dt$. Prove that

$$f'(x) = x\int_0^x g(t)dt - \int_0^x tg(t)dt,$$

then compute $f''(1)$ and $f'''(1)$.

$\boxed{\text{Solution}}$ (4 points) We first observe that

$$f(x) = \frac{1}{2}\left(x^2\int_0^x g(t)dt - 2x\int_0^x tg(t)dt + \int_0^x t^2 g(t)dt\right).$$

This follows simply from the fact that $(x-t)^2 = x^2 - 2xt + t^2$, $x$ is a scalar and the linearity of the integral. Now we differentiate. Here the first term uses the product rule and the fundamental theorem of calculus, the second term uses the same, and the third term requires only the fundamental theorem of calculus:

$$\frac{d}{dx}\left(x^2\int_0^x g(t)dt - 2x\int_0^x tg(t)dt + \int_0^x t^2 g(t)dt\right)$$

$$= x^2 g(x) + 2x\int_0^x g(t)dt - 2\int_0^x tg(t)dt - 2x^2 g(x) + x^2 g(x)$$

$$= 2x\int_0^x g(t)dt - 2\int_0^x tg(t)dt.$$

Dividing the result by 2 answers the first part of the question. Now, observe by the Fundamental Theorem of Calculus (and the product rule)

$$f''(x) = \int_0^x g(t)dt + xg(x) - xg(x) = \int_0^x g(t)dt.$$

Thus, f"(1)=2. Moreover, $f'''(x) = g(x)$ so $f'''(1) = g(1) = 5$.

**Problem 2:**(a) Let $f(x) = e^x - 1 - x$ for all $x$. Prove that $f'(x) \geq 0$ if $x \geq 0$ and $f'(x) \leq 0$ if $x \leq 0$. Use this fact to deduce the inequalities

$$e^x > 1 + x, \quad e^{-x} > 1 - x,$$

for all $x > 0$. Integrate these inequalities to derive the following further inequalities, all valid for $x > 0$;

(b) $e^x > 1 + x + x^2/2!$,    $e^{-x} < 1 - x + x^2/2!$.

(c) $e^x > 1 + x + x^2/2! + x^3/3!$,    $e^{-x} < 1 - x + x^2/2! - x^3/3!$.

(d) Guess the generalization and prove your result.

$\boxed{\text{Solution}}$ (4 points) (a) First observe that $f'(x) = e^x - 1$. By Theorem 1 of Course Notes M, we know that $e^0 = 1$, so $f'(0) = 0$. Moreover, Theorem 2 tells us that $f''(x) = e^x$ is positive. By the Mean Value Theorem, $f'$ is strictly increasing. Since $f'(0) = 0$, we see that $f'(x) \geq 0$ for $x \geq 0$ and $f'(x) \leq 0$ for $x \leq 0$.

Now, $f'(x) \geq 0$ implies the function $f$ is increasing for $x \geq 0$. That is, $f(x) = e^x - 1 - x > f(0) = 0$. So, for $x > 0$, $e^x > 1 + x$. Now, as $f'(x) \leq 0$ for $x \leq 0$, we see that for negative $x$ the function is decreasing. (This is again by the MVT.) Let $y > 0$. Then $-y < 0$ and thus $f(-y) = e^{-y} - 1 - (-y) > f(0) = 0$. That is, $e^{-y} > 1 - y$ for $y > 0$.

(b) First consider the inequality $e^t > 1 + t$ for $t > 0$. We integrate both sides of this inequality from 0 to $x$. Then,

$$\int_0^x e^t dt = e^x - e^0 = e^x - 1 > \int_0^x 1 + t \, dt = x + x^2/2$$

where the comparison theorem gives the inequality. Thus, $e^x > 1 + x + x^2/2$.

Now consider the second inequality. In this case, we again integrate from 0 to $x$ the inequality $e^{-t} > 1 - t$. Here we see

$$\int_0^x e^{-t} dt = -(e^{-x} - 1) > \int_0^x 1 - t \, dt = x - x^2/2.$$

Simplifying the expression yields $e^{-x} < 1 - x + x^2/2$.

(c) We proceed in each case exactly as in the previous part of the problem. In particular,

$$\int_0^x e^t dt = e^x - 1 > \int_0^x 1 + t + t^2/2 \, dt = x + x^2/2 + x^3/(3 \cdot 2) = x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

which is exactly what we want. In the second case, the inequality sign will switch directions again because of the negative sign that shows up in the process of integration. That is

$$\int_0^x e^{-t} dt = -(e^{-x} - 1) < \int_0^x 1 - t + t^2/2 \, dt = x - x^2/2 + x^3/(3 \cdot 2).$$

2

Again, quickly simplifying the expression we get exactly as expected.

(d) Finally, the generalization appears to be the following:

$$e^x > \sum_{k=0}^{n} \frac{x^k}{k!}$$

and

$$\sum_{k=0}^{2n+1} (-1)^k \frac{x^k}{k!} < e^{-x} < \sum_{k=0}^{2n} (-1)^k \frac{x^k}{k!}.$$

for all $x > 0$ and $n \in \mathbb{Z}^+$.

We prove each of these by induction on $n$. For each situation, we already know the inequalities hold for $n = 0, 1$. So, we assume now the inequalities hold for some fixed $n \in \mathbb{Z}^+$ and we show this implies they both hold for $n+1$. First, we use the linearity of the integral, the rule for polynomial integration, and the induction hypothesis to see

$$\int_0^x e^t dt = e^x - 1 > \int_0^x \sum_{k=0}^{n} \frac{t^k}{k!} dt = \sum_{k=0}^{n} \int_0^x \frac{t^k}{k!} dt = \sum_{k=0}^{n} \frac{x^{k+1}}{(k+1)k!} = \sum_{k=1}^{n+1} \frac{x^k}{k!}.$$

Adding 1 to both sides of the inequality gives the result.

Using the same technique on the second string of inequalities, first observe that

$$\int_0^x \sum_{k=0}^{2n+1} (-1)^k \frac{t^k}{k!} dt = \sum_{k=0}^{2n+1} (-1)^k \int_0^x \frac{t^k}{k!} dt$$

$$= \sum_{k=0}^{2n+1} (-1)^k \frac{x^{k+1}}{(k+1) \cdot k!} = \sum_{k=1}^{2n+2} (-1)^{k+1} \frac{x^k}{k!}$$

$$< \int_0^x e^{-t} dt = -(e^{-x} - 1)$$

To simplify the expression, we multiply the inequality by $-1$, which switches the direction of the inequality to get

$$e^{-x} - 1 < \sum_{k=1}^{2n+2} (-1)^k \frac{x^k}{k!}.$$

Adding 1 to both sides gives

$$e^{-x} < \sum_{k=0}^{2n+2} (-1)^k \frac{x^k}{k!} = \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^k}{k!}.$$

3

This is one result. To get the second, we perform the integration on the inequality just above. That is

$$\int_0^x \sum_{k=0}^{2(n+1)} (-1)^k \frac{t^k}{k!} dt = \sum_{k=0}^{2(n+1)} (-1)^k \int_0^x \frac{t^k}{k!} dt$$

$$= \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^{k+1}}{(k+1) \cdot k!} = \sum_{k=1}^{2(n+1)+1} (-1)^{k+1} \frac{x^k}{k!}$$

$$> \int_0^x e^{-t} dt = -(e^{-x} - 1).$$

Again, multiplying both sides of the inequality by $-1$ and then adding 1 gives the result. That is

$$\sum_{k=0}^{2(n+1)+1} (-1)^k \frac{x^k}{k!} < e^{-x} < \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^k}{k!}.$$

**Problem 3:** (a) Find the derivative of $f(x) = x(\sin(\log(x)) - \cos(\log(x)))$.

(b) Find the derivative of $f(x) = x^{a^a} + a^{x^a} + a^{a^x}$.

$\boxed{\text{Solution}}$ (4 points) (a) We use primarily the product and chain rule to conclude

$$f'(x) = \sin(\log x) - \cos(\log x) + x \frac{d}{dx}(\sin(\log x) - \cos(\log x))$$

$$= \sin(\log x) - \cos(\log x) + x \cos(\log x) \cdot \frac{1}{x} - x(-\sin(\log x)) \cdot \frac{1}{x}$$

$$= \sin(\log x) - \cos(\log x) + \cos(\log x) + \sin(\log x)$$

$$= 2\sin(\log x).$$

(b) We treat each term separately, and make frequent use of Theorem 6 from Course Notes M. First, we can simply use the power rule on the first term:

$$\frac{d}{dx} x^{a^a} = a^a (x^{a^a - 1}).$$

For the final two terms, we observe that if $g(x) = a^x$, then $g(x^a) = a^{x^a}$ and $g(a^x) = a^{a^x}$. So for both of these, we need to use the chain rule. Here

$$\frac{d}{dx} g(x^a) = g'(x^a) \frac{d}{dx}(x^a) = \log a \cdot a^{x^a} \cdot a x^{a-1}$$

4

and
$$\frac{d}{dx}g(a^x) = g'(a^x)\frac{d}{dx}(a^x) = \log a \cdot a^{a^x} \log a \cdot a^x = (\log a)^2 a^x a^{a^x}.$$

Putting it all together,

$$f'(x) = a^a x^{a^a-1} + a\log(a)a^{x^a}x^{a-1} + (\log a)^2 a^x a^{a^x}.$$

**Bonus:** Prove that if $f'$ is increasing then $f$ intersects each tangent line exactly once.

Solution (4 points) We proceed by contradiction and suppose there exists some tangent line that intersects $f$ more than once. Suppose the tangent line is at the point $(x_0, f(x_0))$. Then it is of the form $g(x) = f(x_0)+f'(x_0)(x-x_0)$. By hypothesis, $h(x) = g(x) - f(x)$ has at least two zeros. Call the second zero $x_1$. Then $h(x_0) = h(x_1) = 0$. By the Mean Value Theorem, as $h$ is differentiable, there exists some $c$ in the interval with endpoints $x_0, x_1$ such that $h'(c) = 0$. Now, $h'(x) = f'(x_0) - f'(x)$ and so $f'(c) = f'(x_0)$. But, as $c \neq x_0$, this implies that $f'$ is not increasing. This gives the necessary contradiction.

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