18.01 Single Variable Calculus
Fall 2006

Lecture 26: Trigonometric Integrals and Substitution

Trigonometric Integrals

How do you integrate an expression like $\int \sin^n x \cos^m x \, dx$? ($n = 0, 1, 2...$ and $m = 0, 1, 2, \ldots$)

We already know that:

$$\int \sin x \, dx = -\cos x + c \quad \text{and} \quad \int \cos x \, dx = \sin x + c$$

Method A

Suppose either n or m is odd.

Example 1. $\int \sin^3 x \cos^2 x \, dx$.
Our strategy is to use $\sin^2 x + \cos^2 x = 1$ to rewrite our integral in the form:

$$\int \sin^3 x \cos^2 x \, dx = \int f(\cos x) \, \sin x \, dx$$

Indeed,

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \, \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \, \sin x \, dx$$

Next, use the substitution

$$u = \cos x \quad \text{and} \quad du = -\sin x \, dx$$

Then,

$$\int (1 - \cos^2 x) \, \cos^2 x \, \sin x \, dx = \int (1 - u^2) u^2 (-du)$$

$$= \int (-u^2 + u^4) du = -\frac{1}{3} u^3 + \frac{1}{5} u^5 + c = -\frac{1}{3} \cos^3 u + \frac{1}{5} \cos^5 x + c$$

Example 2.

$$\int \cos^3 x \, dx = \int f(\sin x) \, \cos x \, dx = \int (1 - \sin^2 x) \, \cos x \, dx$$

Again, use a substitution, namely

$$u = \sin x \quad \text{and} \quad du = \cos x \, dx$$

$$\int \cos^3 x \, dx = \int (1 - u^2) du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c$$

Method B

This method requires *both* m and n to be even. It requires double-angle formulae such as

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

(Recall that $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\sin^2 x) = 2\cos^2 x - 1$)
Integrating gets us

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \frac{x}{2} + \frac{\sin(2x)}{4} + c$$

We follow a similar process for integrating $\sin^2 x$.

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}dx = \frac{x}{2} - \frac{\sin(2x)}{4} + c$$

The full strategy for these types of problems is to keep applying Method B until you can apply Method A (when one of m or n is odd).

Example 3. $\int \sin^2 x \cos^2 x\,dx$.
Applying Method B twice yields

$$\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \int \left(\frac{1}{4} - \frac{1}{4}\cos^2 2x\right)dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{8}(1+\cos 4x)\right)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

There is a shortcut for Example 3. Because $\sin 2x = 2\sin x\cos x$,

$$\int \sin^2 x\cos^2 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx = \frac{1}{4}\int \frac{1-\cos 4x}{2}dx = \text{ same as above}$$

The next family of trig integrals, which we'll start today, but will not finish is:

$$\int \sec^n x\,\tan^m x\,dx \quad \text{where } n = 0,1,2,\ldots \text{ and } m = 0,1,2,\ldots$$

Remember that

$$\sec^2 x = 1 + \tan^2 x$$

which we double check by writing

$$\frac{1}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^3 x}$$

$$\boxed{\int \sec^2 x\,dx = \tan x + c \qquad \int \sec x\,\tan x\,dx = \sec x + c}$$

To calculate the integral of $\tan x$, write

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$

Let $u = \cos x$ and $du = -\sin x\,dx$, then

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u} = -\ln(u) + c$$

$$\boxed{\int \tan x\,dx = -\ln(\cos x) + c}$$

(We'll figure out what $\int \sec x\,dx$ is later.)

Now, let's see what happens when you have an even power of secant. (The case n even.)

$$\int \sec^4 x\,dx = \int f(\tan x)\,\sec^2 x\,dx = \int (1+\tan^2 x)\,\sec^2 x\,dx$$

Make the following substitution:

$$u = \tan x$$

and

$$du = \sec^2 x\,dx$$

$$\int \sec^4 x\,dx = \int (1+u^2)du = u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c$$

What happens when you have a odd power of tan? (The case m odd.)

$$\int \tan^3 x\,\sec x\,dx = \int f(\sec x)\,d(\sec x)$$

$$= \int (\sec^2 x - 1)\,\sec x\,\tan x\,dx$$

(Remember that $\sec^2 x - 1 = \tan^2 x$.)

Use substitution:

$$u = \sec x$$

and

$$du = \sec x\,\tan x\,dx$$

Then,

$$\int \tan^3 x\,\sec x\,dx = \int (u^2-1)du = \frac{u^3}{3} - u + c = \frac{\sec^3 x}{3} - \sec x + c$$

We carry out one final case: $n = 1, m = 0$

$$\int \sec x\,dx = \ln(\tan x + \sec x) + c$$

We get the answer by "advanced guessing," i.e., "knowing the answer ahead of time."

$$\int \sec x \, dx = \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx$$

Make the following substitutions:

$$u = \tan x + \sec x$$

and

$$du = (\sec^2 x + \sec x \tan x) \, dx$$

This gives

$$\int \sec x \, dx = \int \frac{du}{u} = \ln(u) + c = \ln(\tan x + \sec x) + c$$

Cases like $n = 3, m = 0$ or more generally n odd and m even are more complicated and will be discussed later.

Trigonometric Substitution

Knowing how to evaluate all of these trigonometric integrals turns out to be useful for evaluating integrals involving square roots.

Example 4. $y = \sqrt{a^2 - x^2}$

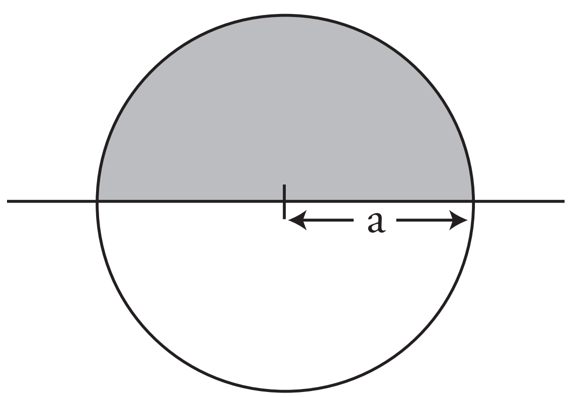

Figure 1: Graph of the circle $x^2 + y^2 = a^2$.

We already know that the area of the top half of the disk is

$$\int_{-a}^{a} \sqrt{a^2 - x^2} \, dx = \frac{\pi a^2}{2}$$

What if we want to find this area?

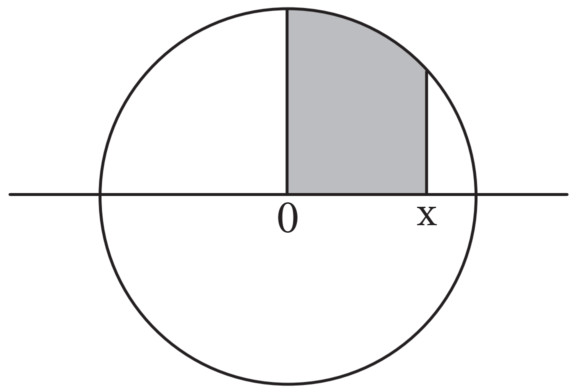

Figure 2: Area to be evaluated is shaded.

To do so, you need to evaluate this integral:

$$\int_{t=0}^{t=x} \sqrt{a^2 - t^2}\, dt$$

Let $t = a \sin u$ and $dt = a \cos u\, du$. (Remember to change the limits of integration when you do a change of variables.)
Then,

$$a^2 - t^2 = a^2 - a^2 \sin^2 u = a^2 \cos^2 u; \quad \sqrt{a^2 - t^2} = a \cos u$$

Plugging this into the integral gives us

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \int (a \cos u)\, a \cos u\, du = a^2 \int_{u=0}^{u=\sin^{-1}(x/a)} \cos^2 u\, du$$

Here's how we calculated the new limits of integration:

$$\begin{aligned} t &= 0 \implies a \sin u = 0 \implies u = 0 \\ t &= x \implies a \sin u = x \implies u = \sin^{-1}(x/a) \end{aligned}$$

$$\int_0^x \sqrt{a^2 - t^2}\, dt = a^2 \int_0^{\sin^{-1}(x/a)} \cos^2 u\, du = a^2 \left(\frac{u}{2} + \frac{\sin 2u}{4} \right) \Big|_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2 \sin^{-1}(x/a)}{2} + \left(\frac{a^2}{4} \right) \left(2 \sin(\sin^{-1}(x/a)) \cos(\sin^{-1}(x/a)) \right)$$

(Remember, $\sin 2u = 2 \sin u\, \cos u$.)

We'll pick up from here next lecture (Lecture 28 since Lecture 27 is Exam 3).

Lecture 28: Integration by Inverse Substitution; Completing the Square

Trigonometric Substitutions, continued

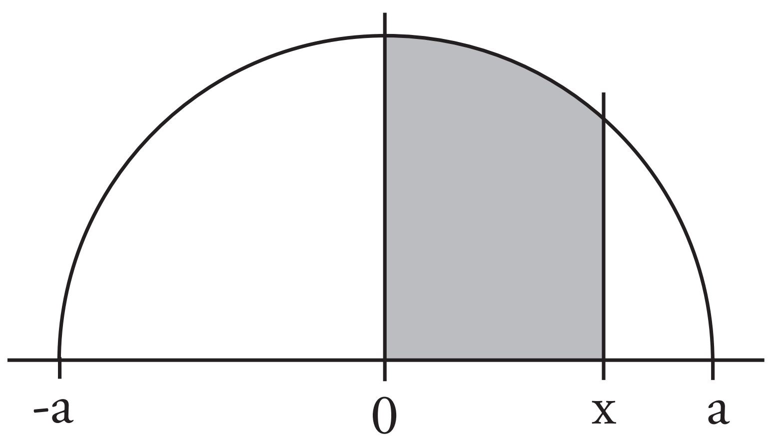

Figure 1: Find area of shaded portion of semicircle.

$$\int_0^x \sqrt{a^2 - t^2}\,dt$$

$$t = a \sin u; \quad dt = a \cos u\, du$$

$$a^2 - t^2 = a^2 - a^2 \sin^2 u = a^2 \cos^2 u \implies \sqrt{a^2 - t^2} = a \cos u \quad \text{(No more square root!)}$$

Start: $x = -a \Leftrightarrow u = -\pi/2$; Finish: $x = a \Leftrightarrow u = \pi/2$

$$\int \sqrt{a^2 - t^2}\,dt = \int a^2 \cos^2 u\, du = a^2 \int \frac{1 + \cos(2u)}{2} du = a^2 \left[\frac{u}{2} + \frac{\sin(2u)}{4}\right] + c$$

$$\text{(Recall, } \cos^2 u = \frac{1 + \cos(2u)}{2}\text{).}$$

We want to express this in terms of x, not u. When $t = 0$, $a \sin u = 0$, and therefore $u = 0$. When $t = x$, $a \sin u = x$, and therefore $u = \sin^{-1}(x/a)$.

$$\frac{\sin(2u)}{4} = \frac{2 \sin u \cos u}{4} = \frac{1}{2} \sin u \cos u$$

$$\sin u = \sin\left(\sin^{-1}(x/a)\right) = \frac{x}{a}$$

How can we find $\cos u = \cos\left(\sin^{-1}(x/a)\right)$? Answer: use a right triangle (Figure 2).

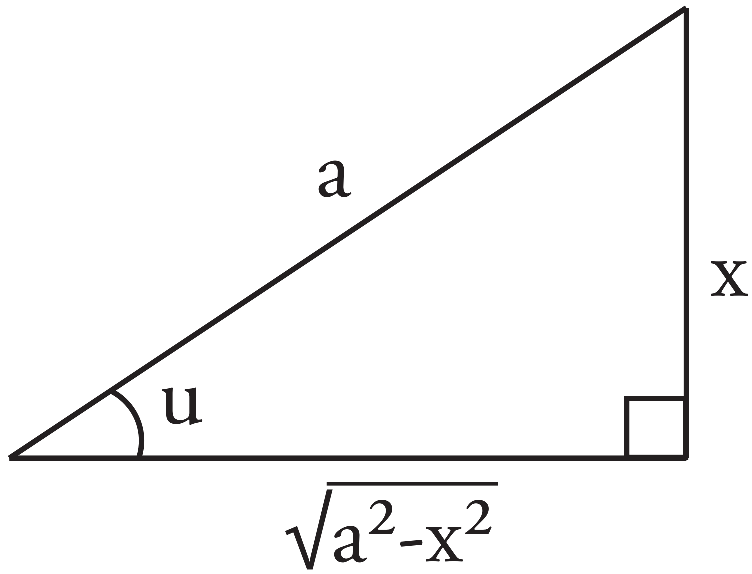

Figure 2: $\sin u = x/a$; $\cos u = \sqrt{a^2 - x^2}/a$.

From the diagram, we see

$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$

And finally,

$$\int_0^x \sqrt{a^2 - t^2}\, dt = a^2\left[\frac{u}{4} + \frac{1}{2}\sin u \cos u\right] - 0 = a^2\left[\frac{\sin^{-1}(x/a)}{2} + \frac{1}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a}\right]$$

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2}$$

When the *answer* is this complicated, the route to getting there has to be rather complicated. There's no way to avoid the complexity.

Let's double-check this answer. The area of the upper shaded sector in Figure 3 is $\frac{1}{2}a^2u$. The area of the lower shaded region, which is a triangle of height $\sqrt{a^2 - x^2}$ and base x, is $\frac{1}{2}x\sqrt{a^2 - x^2}$.

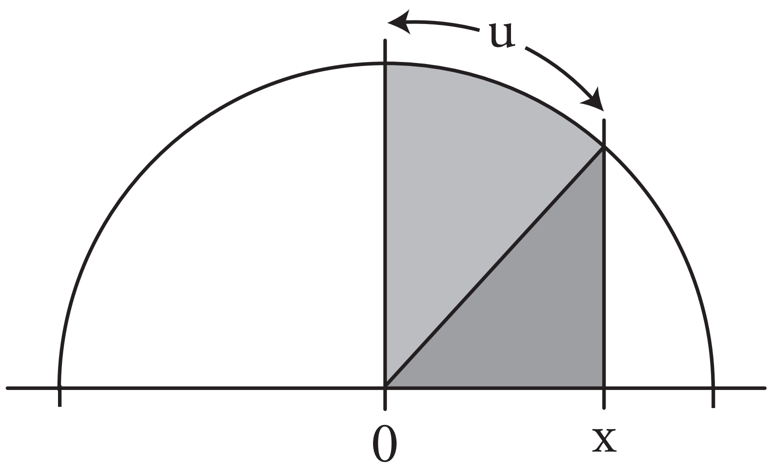

Figure 3: Area divided into a sector and a triangle.

Here is a list of integrals that can be computed using a trig substitution and a trig identity.

integral	substitution	trig identity
$\int \frac{dx}{\sqrt{x^2+1}}$	$x = \tan u$	$\tan^2 u + 1 = \sec^2 u$
$\int \frac{dx}{\sqrt{x^2-1}}$	$x = \sec u$	$\sec^2 u - 1 = \tan^2 u$
$\int \frac{dx}{\sqrt{1-x^2}}$	$x = \sin u$	$1 - \sin^2 u = \cos^2 u$

Let's extend this further. How can we evaluate an integral like this?

$$\int \frac{dx}{\sqrt{x^2+4x}}$$

When you have a linear and a quadratic term under the square root, complete the square.

$$x^2 + 4x = (\text{something})^2 \pm \text{constant}$$

In this case,

$$(x+2)^2 = x^2 + 4x + 4 \implies x^2 + 4x = (x+2)^2 - 4$$

Now, we make a substitution.

$$v = x + 2 \quad \text{and} \quad dv = dx$$

Plugging these in gives us

$$\int \frac{dx}{\sqrt{(x+2)^2 - 4}} = \int \frac{dv}{\sqrt{v^2 - 4}}$$

Now, let

$$v = 2\sec u \quad \text{and} \quad dv = 2\sec u \tan u$$

$$\int \frac{dv}{\sqrt{v^2-4}} = \int \frac{2\sec u\, \tan u\, du}{2\tan u} = \int \sec u\, du$$

Remember that
$$\int \sec u \, du = \ln(\sec u + \tan u) + c$$
Finally, rewrite everything in terms of x.
$$v = 2\sec u \Leftrightarrow \cos u = \frac{2}{v}$$
Set up a right triangle as in Figure 4. Express $\tan u$ in terms of v.

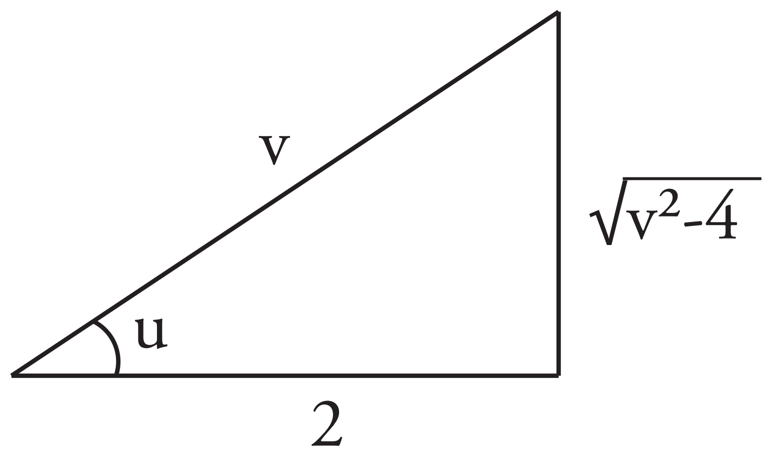

Figure 4: $\sec u = v/2 \quad \text{or} \quad \cos u = 2/v.$

Just from looking at the triangle, we can read off
$$\sec u = \frac{v}{2} \quad \text{and} \quad \tan u = \frac{\sqrt{v^2-4}}{2}$$
$$\int 2\sec u \, du = \ln\left(\frac{v}{2} + \frac{\sqrt{v^2-4}}{2}\right) + c$$
$$= \ln(v + \sqrt{v^2-4}) - \ln 2 + c$$
We can combine those last two terms into another constant, $\tilde{c}$.
$$\int \frac{dx}{\sqrt{x^2+4x}} = \ln(x + 2 + \sqrt{x^2+4x}) + \tilde{c}$$

Here's a teaser for next time. In the next lecture, we'll integrate all rational functions. By "rational functions," we mean functions that are the ratios of polynomials:
$$\frac{P(x)}{Q(x)}$$

It's easy to evaluate an expression like this:
$$\int \left(\frac{1}{x-1} + \frac{3}{x+2}\right) dx = \ln|x-1| + 3\ln|x+2| + c$$

If we write it a bit differently, however, it becomes much harder to integrate:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{x+2+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

$$\int \frac{4x-1}{x^2+x-2} = ???$$

How can we reorganize what to do starting from $(4x-1)/(x^2+x-2)$? Next time, we'll see how. It involves some algebra.

Lecture 29: Partial Fractions

We continue the discussion we started last lecture about integrating rational functions. We defined a rational function as the ratio of two polynomials:

$$\frac{P(x)}{Q(x)}$$

We looked at the example

$$\int \left[\frac{1}{x-1} + \frac{3}{x+2}\right] dx = \ln|x-1| + 3\ln|x+2| + c$$

That same problem can be disguised:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{(x+2)+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

which leaves us to integrate this:

$$\int \frac{4x-1}{x^2+x-2}\, dx = ???$$

Goal: we want to figure out a systematic way to split $\frac{P(x)}{Q(x)}$ into simpler pieces.

First, we factor the denominator $Q(x)$.

$$\frac{4x-1}{x^2+x-2} = \frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

There's a slow way to find A and B. You can clear the denominator by multiplying through by $(x-1)(x+2)$:

$$(4x-1) = A(x+2) + B(x-1)$$

From this, you find

$$4 = A + B \quad \text{and} \quad -1 = 2A - B$$

You can then solve these simultaneous linear equations for A and B. This approach can take a very long time if you're working with 3, 4, or more variables.

There's a faster way, which we call the "cover-up method". Multiply both sides by $(x-1)$:

$$\frac{4x-1}{x+2} = A + \frac{B}{x+2}(x-1)$$

Set $x = 1$ to make the B term drop out:

$$\frac{4-1}{1+2} = A$$

$$A = 1$$

The fastest way is to do this in your head or physically *cover up* the struck-through terms. For instance, to evaluate B:

$$\frac{4x-1}{(x-1)\cancel{(x+2)}} = \cancel{\frac{A}{x-1}} + \frac{B}{\cancel{(x+2)}}$$

Implicitly, we are multiplying by $(x+2)$ and setting $x=-2$. This gives us

$$\frac{4(-2)-1}{-2-1} = B \quad \Longrightarrow \quad B = 3$$

What we've described so far works when $Q(x)$ factors completely into *distinct* factors and the degree of P is less than the degree of Q.

If the factors of Q repeat, we use a slightly different approach. For example:

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Use the cover-up method on the highest degree term in $(x-1)$.

$$\frac{x^2+1}{x+2} = B + [\text{stuff}](x-1)^2 \quad \Longrightarrow \quad \frac{1^2+2}{1+2} = B \quad \Longrightarrow \quad B = 1$$

Implicitly, we multiplied by $(x-1)^2$, then took the limit as $x \to 1$.

C can also be evaluated by the cover-up method. Set $x=-2$ to get

$$\frac{x^2+2}{(x-1)}^2 = C + [\text{stuff}](x+2) \quad \Longrightarrow \quad \frac{(-2)^2+2}{(-2-1)^2} = C \quad \Longrightarrow \quad C = \frac{2}{3}$$

This yields

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{1}{(x-1)^2} + \frac{2/3}{x+2}$$

Cover-up can't be used to evaluate A. Instead, plug in an easy value of x: $x=0$.

$$\frac{2}{(-1)^2(2)} = \frac{A}{-1} + 1 + \frac{1}{3} \Longrightarrow 1 = 1 + \frac{1}{3} - A \Longrightarrow A = \frac{1}{3}$$

Now we have a complete answer:

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{1}{3(x-1)} + \frac{1}{(x-1)^2} + \frac{2}{3(x+2)}$$

Not all polynomials factor completely (without resorting to using complex numbers). For example:

$$\frac{1}{(x^2+1)(x-1)} = \frac{A_1}{x-1} + \frac{B_1x+C_1}{x^2+1}$$

We find A_1, as usual, by the cover-up method.

$$\frac{1}{1^2+1} = A_1 \quad \Longrightarrow \quad A_1 = \frac{1}{2}$$

Now, we have

$$\frac{1}{(x^2+1)(x-1)} = \frac{1/2}{x-1} + \frac{B_1 x + C_1}{x^2+1}$$

Plug in $x = 0$.

$$\frac{1}{1(-1)} = -\frac{1}{2} + \frac{C_1}{1} \quad \Longrightarrow \quad C_1 = -\frac{1}{2}$$

Now, plug in any value other than $x = 0, 1$. For example, let's use $x = -1$.

$$\frac{1}{2(-2)} = \frac{1/2}{-2} + \frac{B_1(-1) - 1/2}{2} \Longrightarrow 0 = -\frac{B_1 - 1/2}{2} \Longrightarrow B_1 = -\frac{1}{2}$$

Alternatively, you can multiply out to clear the denominators (not done here).

Let's try to integrate this function, now.

$$\int \frac{dx}{(x^2+1)(x-1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|\,x^2+1\,| - \frac{1}{2}\tan^{-1}x + c$$

What if we're faced with something that looks like this?

$$\int \frac{dx}{(x-1)^{10}}$$

This is actually quite simple to integrate:

$$\int \frac{dx}{(x-1)^{10}} = -\frac{1}{9}(x-1)^{-9} + c$$

What about this?

$$\int \frac{dx}{(x^2+1)^{10}}$$

Here, we would use trig substitution:

$$x = \tan u \quad \text{and} \quad dx = \sec^2 u du$$

and the trig identity

$$\tan^2 u + 1 = \sec^2 u$$

to get

$$\int \frac{\sec^2 u\,du}{(\sec^2 u)^{10}} = \int \cos^{18} u\,du$$

From here, we can evaluate this integral using the methods we introduced two lectures ago.

Lecture 30: Integration by Parts, Reduction Formulae

Integration by Parts

Remember the product rule:

$$(uv)' = u'v + uv'$$

We can rewrite that as

$$uv' = (uv)' - u'v$$

Integrate this to get the formula for integration by parts:

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example 1. $\int \tan^{-1} x\,dx$.

At first, it's not clear how integration by parts helps. Write

$$\int \tan^{-1} x\,dx = \int \tan^{-1} x(1 \cdot dx) = \int uv'\,dx$$

with

$$u = \tan^{-1} x \quad \text{and} \quad v' = 1.$$

Therefore,

$$v = x \quad \text{and} \quad u' = \frac{1}{1+x^2}$$

Plug all of these into the formula for integration by parts to get:

$$\int \tan^{-1} x\,dx = \int uv'\,dx = (\tan^{-1} x)x - \int \frac{1}{1+x^2}(x)dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

Alternative Approach to Integration by Parts

As above, the product rule:

$$(uv)' = u'v + uv'$$

can be rewritten as

$$uv' = (uv)' - u'v$$

This time, let's take the *definite* integral:

$$\int_a^b uv'\,dx = \int_a^b (uv)'\,dx - \int_a^b u'v\,dx$$

By the fundamental theorem of calculus, we can say

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

Another notation in the indefinite case is

$$\int u\,dv = uv - \int v\,du$$

This is the same because

$$dv = v'\,dx \implies uv'\,dx = u\,dv \quad \text{and} \quad du = u'\,dx \implies u'v\,dx = vu'\,dx = v\,du$$

Example 2. $\int (\ln x)dx$

$$u = \ln x;\ du = \frac{1}{x}dx \quad \text{and} \quad dv = dx;\ v = x$$

$$\int (\ln x)dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + c$$

We can also use "advanced guessing" to solve this problem. We know that the derivative of *something* equals $\ln x$:

$$\frac{d}{dx}(??) = \ln x$$

Let's try

$$\frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

That's almost it, but not quite. Let's repair this guess to get:

$$\frac{d}{dx}\left(x\ln x - x\right) = \ln x + 1 - 1 = \ln x$$

Reduction Formulas (Recurrence Formulas)

Example 3. $\int (\ln x)^n\,dx$

Let's try:

$$u = (\ln x)^n \implies u' = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

$$v' = dx;\ v = x$$

Plugging these into the formula for integration by parts gives us:

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1}\cancelto{1}{x\left(\frac{1}{x}\right)}dx$$

Keep repeating integration by parts to get the full formula: $n \to (n-1) \to (n-2) \to (n-3) \to$ etc

Example 4. $\int x^n e^x\,dx$ Let's try:

$$u = x^n \implies u' = nx^{n-1}; \quad v' = e^x \implies v = e^x$$

Putting these into the integration by parts formula gives us:

$$\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx$$

Repeat, going from $n \to (n-1) \to (n-2) \to$ etc.

Bad news: If you change the integrals just a little bit, they become impossible to evaluate:

$$\int \left(\tan^{-1} x\right)^2 dx = \text{impossible}$$

$$\int \frac{e^x}{x}\,dx = \text{also impossible}$$

Good news: When you can't evaluate an integral, then

$$\int_1^2 \frac{e^x}{x} dx$$

is an *answer*, not a question. This *is* the solution– you don't have to integrate it!

The most important thing is setting up the integral! (Once you've done that, you can always evaluate it numerically on a computer.) So, why bother to evaluate integrals by hand, then? Because you often get families of related integrals, such as

$$F(a) = \int_1^\infty \frac{e^x}{x^a} dx$$

where you want to find how the answer depends on, say, a.

Arc Length

This is very useful to know for 18.02 (multi-variable calculus).

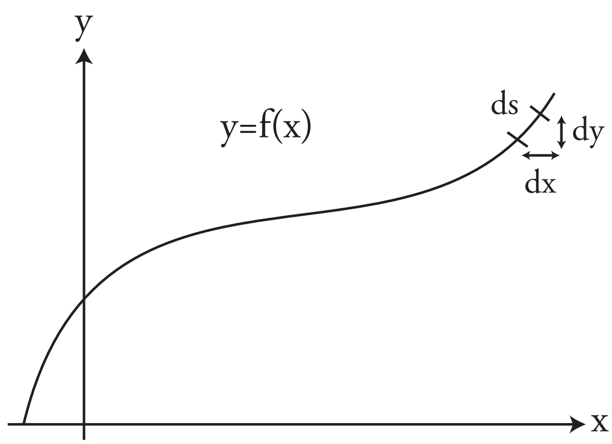

Figure 1: Infinitesimal Arc Length ds

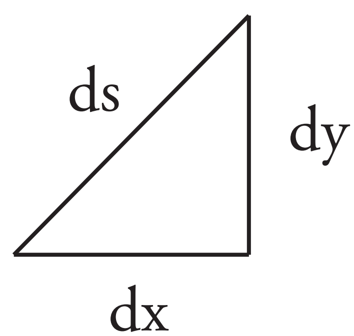

Figure 2: Zoom in on Figure 1 to see an approximate right triangle.

In Figures 1 and 2, s denotes arc length and ds = the infinitesimal of arc length.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\,dx$$

Integrating with respect to ds finds the length of a curve between two points (see Figure 3). To find the length of the curve between P_0 and P_1, evaluate:

$$\int_{P_0}^{P_1} ds$$

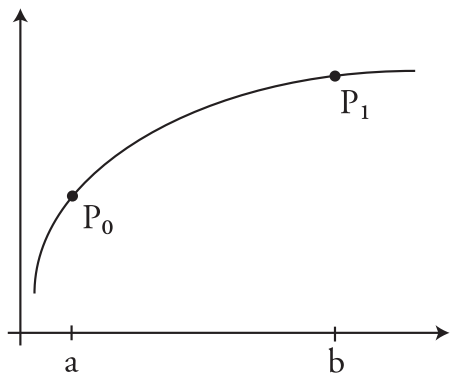

Figure 3: Find length of curve between P_0 and P_1.

We want to integrate with respect to x, not s, so we do the same algebra as above to find ds in terms of dx.

$$\frac{(ds)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

Therefore,

$$\int_{P_0}^{P_1} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 5: The Circle. $x^2 + y^2 = 1$ (see Figure 4).

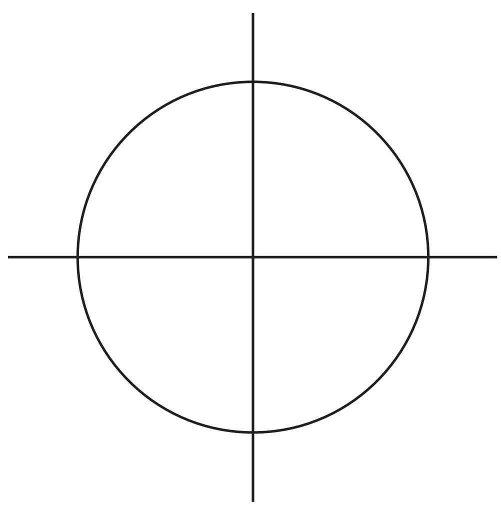

Figure 4: The circle in Example 1.

We want to find the length of the arc in Figure 5:

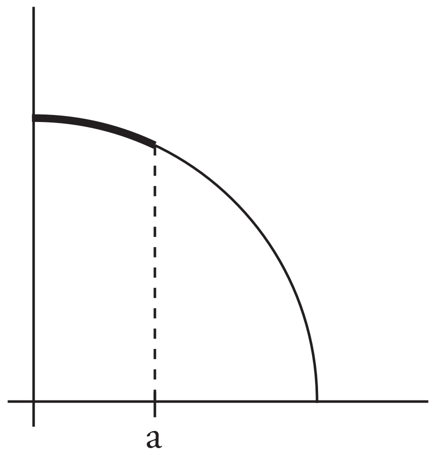

Figure 5: Arc length to be evaluated.

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}\left(\frac{1}{2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1+\frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$ds = \sqrt{\frac{1}{1-x^2}}\,dx$$

$$s = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\,\Big|_0^a = \sin^{-1} a - \sin^{-1} 0 = \sin^{-1} a$$

$$\sin s = a$$

This is illustrated in Figure 6.

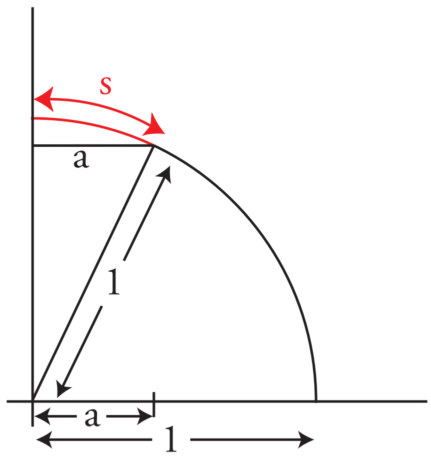

Figure 6: s = angle in radians.

Parametric Equations

Example 6.

$$x = a\cos t$$

$$y = a\sin t$$

Ask yourself: what's constant? What's varying? Here, t is variable and a is constant.
Is there a relationship between x and y? Yes:

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$$

Extra information (besides the circle):
At $t = 0$,

$$x = a\cos 0 = a \quad \text{and} \quad y = a\sin 0 = 0$$

At $t = \frac{\pi}{2}$,

$$x = a\cos\frac{\pi}{2} = 0 \quad \text{and} \quad y = a\sin\frac{\pi}{2} = a$$

Thus, for $0 \le t \le \pi/2$, a quarter circle is traced counter-clockwise (Figure 7).

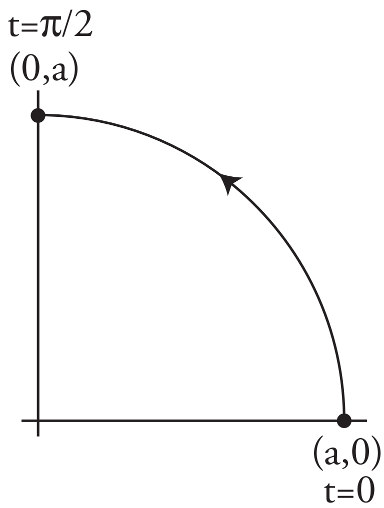

Figure 7: Example 6. $x = a\cos t,\ y = a\sin t;$ the particle is moving counterclockwise.

Example 7: The Ellipse See Figure 8.

$$x = 2\sin t; \quad y = \cos t$$

$$\frac{x^2}{4} + y^2 = 1(\implies (2\sin t)^2/4 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1)$$

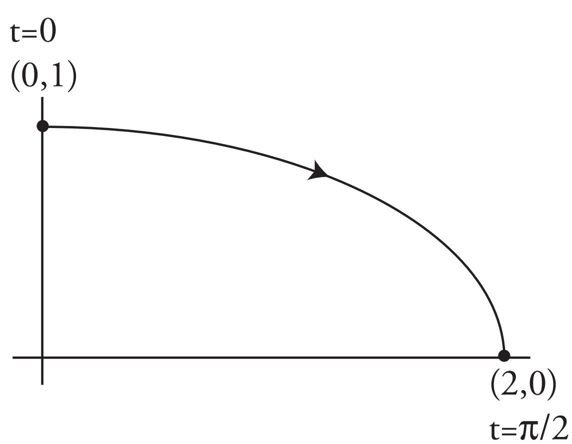

Figure 8: Ellipse: $x = 2\sin t,\ y = \cos t$ (traced clockwise).

Arclength ds for Example 6.

$$dx = -a\sin t\,dt, \quad dy = a\cos t\,dt$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a\sin t\,dt)^2 + (a\cos t\,dt)^2} = \sqrt{(a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$$

Lecture 31: Parametric Equations, Arclength, Surface Area

Arclength, continued

Example 1. Consider this parametric equation:

$$x = t^2 \quad y = t^3 \quad \text{for } 0 \le t \le 1$$

$$x^3 = (t^2)^3 = t^6; \quad y^2 = (t^3)^2 = t^6 \quad \implies x^3 = y^2 \implies y = x^{2/3} \qquad 0 \le x \le 1$$

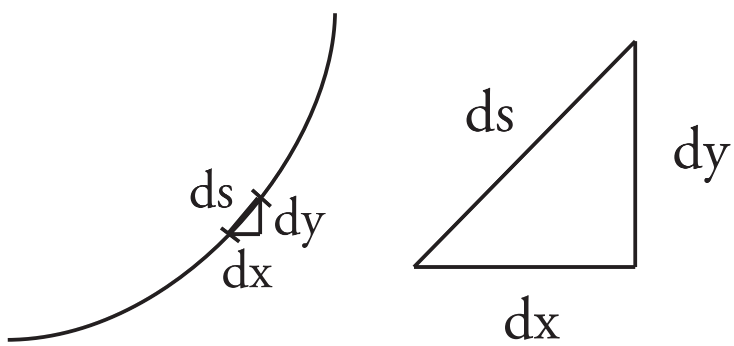

Figure 1: Infinitesimal Arclength.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$(ds)^2 = \underbrace{(2t\,dt)^2}_{(dx)^2} + \underbrace{(3t^2\,dt)^2}_{(dy)^2} = (4t^2 + 9t^4)(dt)^2$$

$$\text{Length} = \int_{t=0}^{t=1} ds = \int_0^1 \sqrt{4t^2 + 9t^4}\,dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt$$

$$= \left.\frac{(4 + 9t^2)^{3/2}}{27}\right|_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2})$$

Even if you can't evaluate the integral analytically, you can always use numerical methods.

Surface Area (surfaces of revolution)

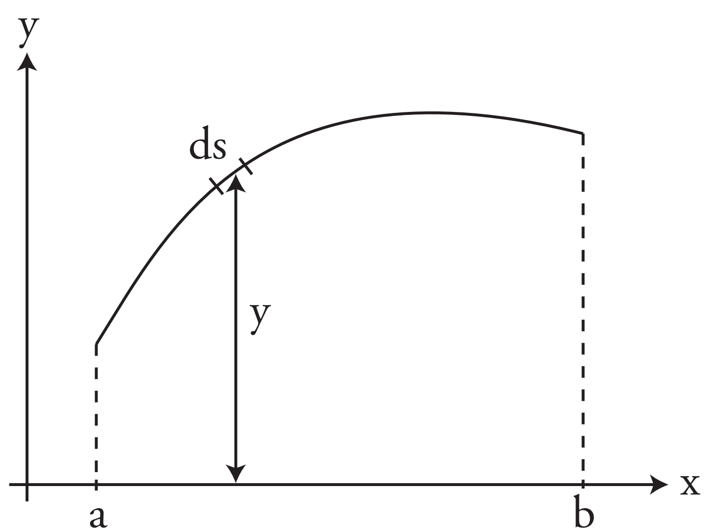

Figure 2: Calculating surface area

ds (the infinitesimal curve length in Figure 2) is revolved a distance $2\pi y$. The surface area of the thin strip of width ds is $2\pi y\, ds$.

Example 2. Revolve Example 1 ($x = t^2, y = t^3, 0 \le t \le 1$) around the x-axis. Refer to Figure 3.

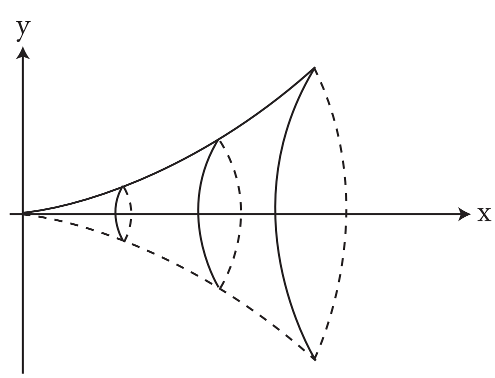

Figure 3: Curved surface of a trumpet.

$$\text{Area } = \int 2\pi y\, ds = \int_0^1 2\pi \underbrace{t^3}_{y} \underbrace{t\sqrt{4+9t^2}\,dt}_{ds} = 2\pi \int_0^1 t^4\sqrt{4+9t^2}\,dt$$

Now, we discuss the method used to evaluate

$$\int t^4(4+9t^2)^{1/2}dt$$

We're going to ignore the factor of 2π. You can reinsert it once you're done evaluating the integral. We use the trigonometric substitution

$$t = \frac{2}{3}\tan u; \quad dt = \frac{2}{3}\sec^2 u\, du; \quad \tan^2 u + 1 = \sec^2 u$$

Putting all of this together gives us:

$$\begin{aligned}\int t^4(4+9t^2)^{1/2}\,dt &= \int \left(\frac{2}{3}\tan u\right)^4 \left(4+9\left(\frac{4}{9}\tan^2 u\right)\right)^{1/2} \left(\frac{2}{3}\sec^2 u\, du\right) \\ &= \left(\frac{2}{3}\right)^5 \int \tan^4 u(2\sec u)(\sec^2 u\, du)\end{aligned}$$

This is a tan − sec integral. It's doable, but it will take a long time for you to work the whole thing out. We're going to stop evaluating it here.

Example 3 Let's use what we've learned to find the surface area of the unit sphere (see Figure 4).

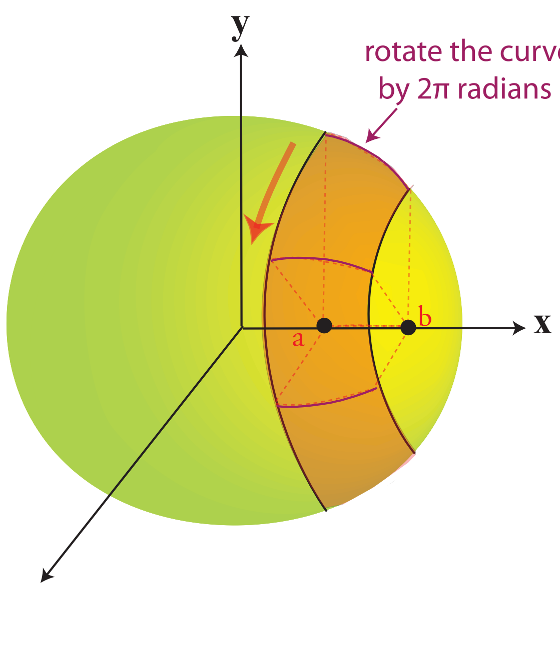

Figure 4: Slice of spherical surface (orange peel, only, not the insides).

For the top half of the sphere,

$$y = \sqrt{1-x^2}$$

We want to find the area of the spherical slice between $x = a$ and $x = b$. A spherical slice has area

$$A = \int_{x=a}^{x=b} 2\pi y \, ds$$

From last time,

$$ds = \frac{dx}{\sqrt{1-x^2}}$$

Plugging that in yields a remarkably simple formula for A:

$$A = \int_a^b 2\pi\sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} = \int_a^b 2\pi \, dx$$

$$= 2\pi(b-a)$$

Special Cases

For a whole sphere, $a = -1$, and $b = 1$.

$$2\pi(1-(-1)) = 4\pi$$

is the surface area of a unit sphere.

For a half sphere, $a = 0$ and $b = 1$.

$$2\pi(1-0) = 2\pi$$

Lecture 32: Polar Co-ordinates, Area in Polar Co-ordinates

Polar Coordinates

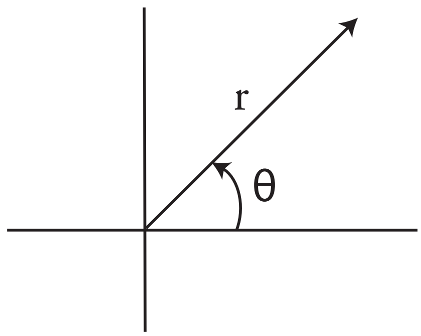

Figure 1: Polar Co-ordinates.

In polar coordinates, we specify an object's position in terms of its distance r from the origin and the angle θ that the ray from the origin to the point makes with respect to the x-axis.

Example 1. What are the polar coordinates for the point specified by $(1, -1)$ in rectangular coordinates?

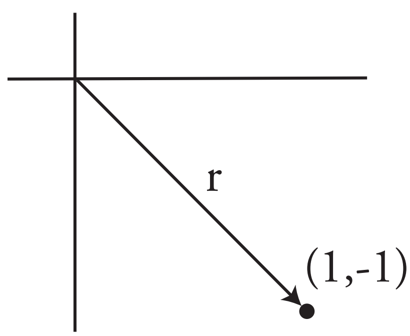

Figure 2: Rectangular Co-ordinates to Polar Co-ordinates.

$$
\begin{aligned}
r &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
\theta &= -\frac{\pi}{4}
\end{aligned}
$$

In most cases, we use the convention that $r \geq 0$ and $0 \leq \theta \leq 2\pi$. But another common convention is to say $r \geq 0$ and $-\pi \leq \theta \leq \pi$. All values of θ and even negative values of r can be used.

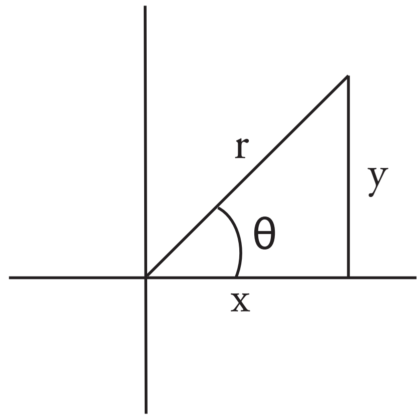

Figure 3: Rectangular Co-ordinates to Polar Co-ordinates.

Regardless of whether we allow positive or negative values of r or θ, what is *always* true is:

$$\boxed{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta}$$

For instance, $x = 1$, $y = -1$ can be represented by $r = -\sqrt{2}$, $\theta = \frac{3\pi}{4}$:

$$1 = x = -\sqrt{2}\cos\frac{3\pi}{4} \quad \text{and} \quad -1 = y = -\sqrt{2}\sin\frac{3\pi}{4}$$

Example 2. Consider a circle of radius a with its center at $x = a$, $y = 0$. We want to find an equation that relates r to θ.

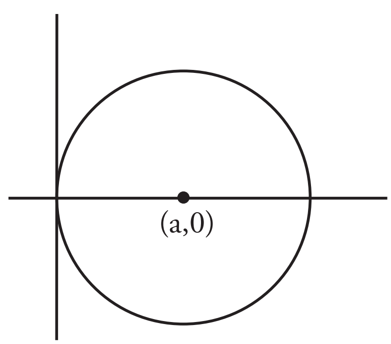

Figure 4: Circle of radius a with center at $x = a, y = 0$.

We know the equation for the circle in rectangular coordinates is

$$(x-a)^2 + y^2 = a^2$$

Start by plugging in:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

This gives us

$$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$$
$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$$
$$r^2 - 2ar\cos\theta = 0$$
$$\boxed{r = 2a\cos\theta}$$

The range of $0 \le \theta \le \dfrac{\pi}{2}$ traces out the top half of the circle, while $-\dfrac{\pi}{2} \le \theta \le 0$ traces out the bottom half. Let's graph this.

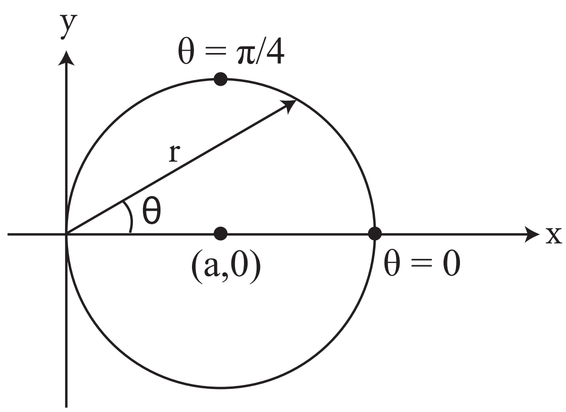

Figure 5: $r = 2a\cos\theta, \quad -\pi/2 \le \theta \le \pi/2.$

At $\theta = 0,\ r = 2a \implies x = 2a,\ y = 0$
At $\theta = \dfrac{\pi}{4},\ r = 2a\cos\dfrac{\pi}{4} = a\sqrt{2}$

The main issue is finding the range of θ tracing the circle once. In this case, $\dfrac{-\pi}{2} < \theta < \dfrac{\pi}{2}$.

$$\theta = -\frac{\pi}{2} \text{ (down)}$$
$$\theta = \frac{\pi}{2} \text{ (up)}$$

Weird range (avoid this one): $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$. When $\theta = \pi$, $r = 2a\cos\pi = 2a(-1) = -2a$. The radius points "backwards". In the range $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$, the same circle is traced out a second time.

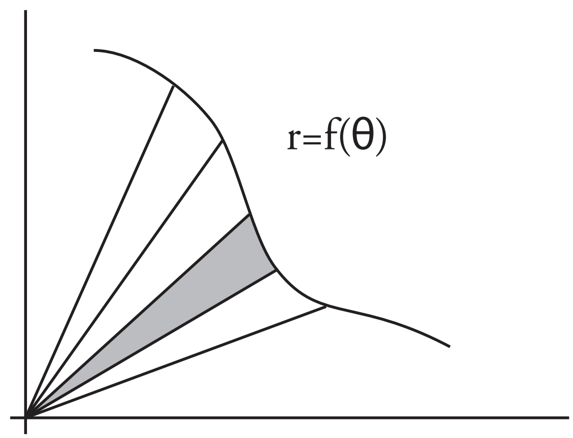

Figure 6: Using polar co-ordinates to find area of a generic function.

Area in Polar Coordinates

Since radius is a function of angle ($r = f(\theta)$), we will integrate with respect to θ. The question is: what, exactly, should we integrate?

$$\int_{\theta_1}^{\theta_2} ??\, d\theta$$

Let's look at a very small slice of this region:

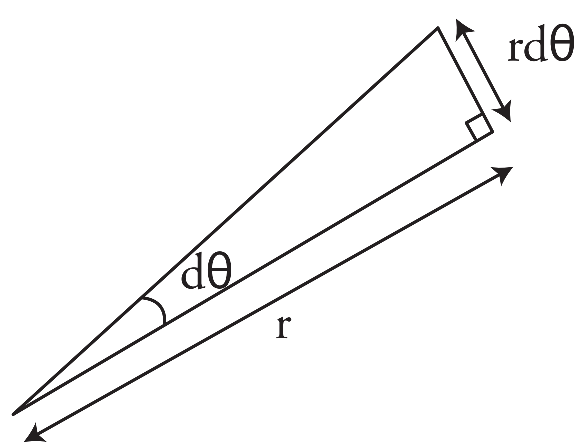

Figure 7: Approximate slice of area in polar coordinates.

This infinitesimal slice is approximately a right triangle. To find its area, we take:

$$\text{Area of slice} \approx \frac{1}{2}(\text{base})\,(\text{height}) = \frac{1}{2}r(r\, d\theta)$$

So,

$$\text{Total Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\, d\theta$$

Example 3. $r = 2a\cos\theta$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (the circle in Figure 5).

$$A = \text{area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(2a\cos\theta)^2\, d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

Because $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, we can rewrite this as

$$A = \text{area} = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta = a^2 \int_{-\pi/2}^{\pi/2} d\theta + a^2 \int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta$$

$$= \pi a^2 + \frac{1}{2}\sin 2\theta \Big|_{-\pi/2}^{\pi/2} = \pi a^2 + \frac{1}{2}\cancelto{0}{[\sin\pi - \sin(-\pi)]}$$

$$A = \text{area} = \pi a^2$$

Example 4: Circle centered at the Origin.

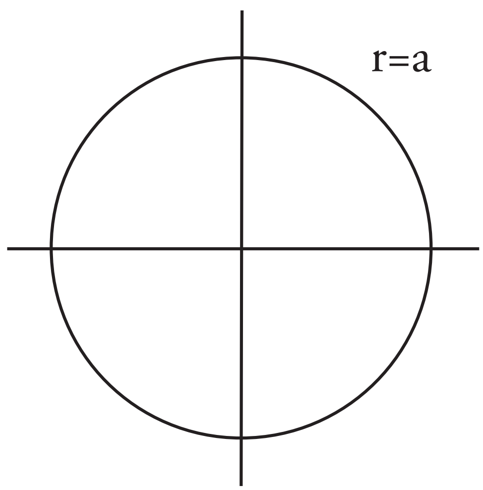

Figure 8: Example 4: Circle centered at the origin

$$x = r\cos\theta;\ y = r\sin\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

The circle is $x^2 + y^2 = a^2$, so $r = a$ and

$$x = a\cos\theta;\ y = a\sin\theta$$

$$A = \int_0^{2\pi} \frac{1}{2}a^2\, d\theta = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2.$$

Example 5: A Ray. In this case, $\theta = b$.

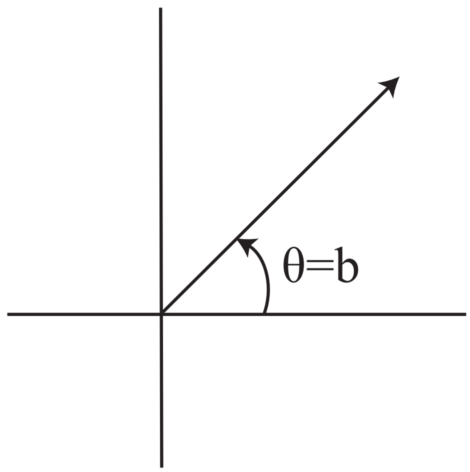

Figure 9: Example 5: The ray $\theta = b, 0 \leq r < \infty$.

The range of r is $0 \leq r < \infty$; $\quad x = r\cos b$; $\quad y = r\sin b$.

Example 6: Finding the Polar Formula, based on the Cartesian Formula

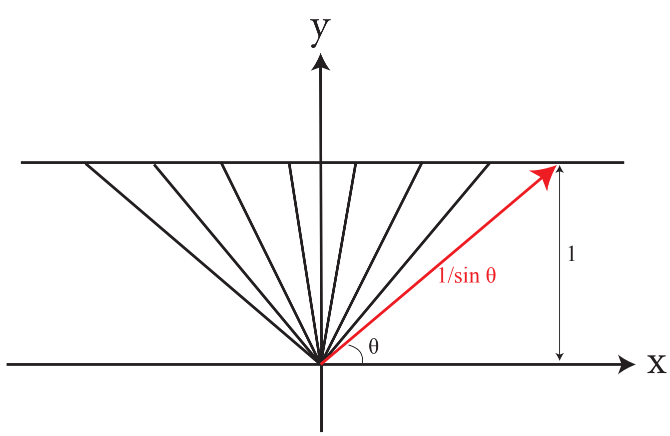

Figure 10: Example 6: Cartesian Form to Polar Form

Consider, in cartesian coordinates, the line $y = 1$. To find the polar coordinate equation, plug in $y = r\sin\theta$ and $x = r\cos\theta$ and solve for r.

$$r\sin\theta = 1 \implies r = \frac{1}{\sin\theta} \quad \text{with} \quad 0 < \theta < \pi$$

Example 7: Going back to (x, y) coordinates from $r = f(\theta)$.
Start with

$$r = \frac{1}{1 + \frac{1}{2}\sin\theta}.$$

Hence,

$$r + \frac{r}{2}\sin\theta = 1$$

Plug in $r = \sqrt{x^2 + y^2}$:

$$\sqrt{x^2+y^2} + \frac{y}{2} = 1$$

$$\sqrt{x^2+y^2} = 1 - \frac{y}{2} \quad \Longrightarrow \quad x^2 + y^2 = \left(1 - \frac{y}{2}\right)^2 = 1 - y + \frac{y^2}{4}$$

Finally,

$$x^2 + \frac{3y^2}{4} + y = 1$$

This is an equation for an ellipse, with the origin at one focus.

Useful conversion formulas:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example 8: A Rose $r = \cos(2\theta)$
The graph looks a bit like a flower:

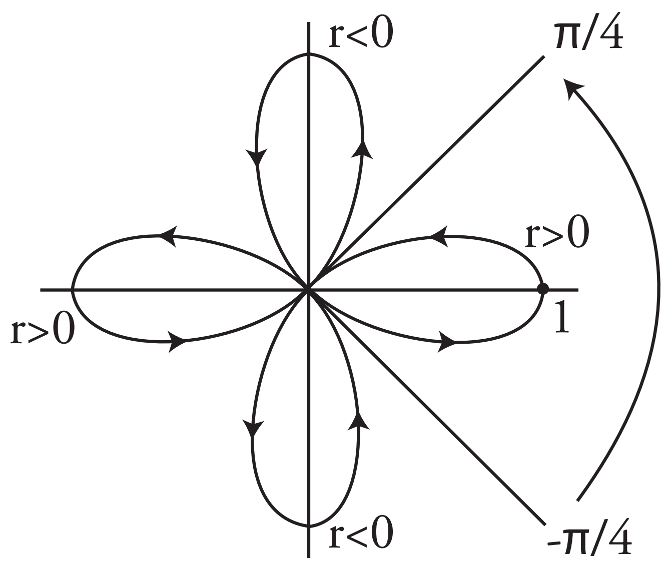

Figure 11: Example 8: Rose

For the first "petal"

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

Note: Next lecture is Lecture 34 as Lecture 33 is Exam 4.

Exam 4 Review

1. Trig substitution and trig integrals.
2. Partial fractions.
3. Integration by parts.
4. Arc length and surface area of revolution
5. Polar coordinates
6. Area in polar coordinates.

Questions from the Students

- Q: What do we need to know about parametric equations?
- A: Just keep this formula in mind:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example: You're given

$$x(t) = t^4$$

and

$$y(t) = 1 + t$$

Find s (length).

$$ds = \sqrt{(4t^3)^2 + (1)^2}dt$$

Then, integrate with respect to t.

- Q: Can you quickly review how to do partial fractions?
- A: When finding partial fractions, first check whether the degree of the numerator is greater than or equal to the degree of the denominator. If so, you first need to do algebraic long-division. If not, then you can split into partial fractions.

Example.

$$\frac{x^2 + x + 1}{(x-1)^2(x+2)}$$

We already know the *form* of the solution:

$$\frac{x^2 + x + 1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

There are two coefficients that are easy to find: B and C. We can find these by the cover-up method.

$$B = \frac{1^2 + 1 + 1}{1+2} = \frac{3}{3} \qquad (x \to 1)$$

To find C,

$$C = \frac{(-2)^2 - 2 + 1}{(-2-1)^2} = \frac{1}{3} \qquad (x \to -2)$$

To find A, one method is to plug in the easiest value of x other than the ones we already used ($x = 1, -2$). Usually, we use $x = 0$.

$$\frac{1}{(-1)^2(2)} = \frac{A}{-1} + \frac{1}{(-1)^2} + \frac{1/3}{2}$$

and then solve to find A.

The Review Sheet handed out during lecture follows on the next page.

Exam 4 Review Handout

1. Integrate by **trigonometric substitution**; evaluate the **trigonometric integral** and work backwards to the original variable by evaluating $\mathbf{trig(trig^{-1})}$ using a right triangle:

a) $a^2 - x^2$ use $x = a\sin u$, $dx = a\cos u\,du$.

b) $a^2 + x^2$ use $x = a\tan u$, $dx = a\sec^2 u\,du$

c) $x^2 - a^2$ use $x = a\sec u$, $dx = a\sec u\tan u\,du$

2. Integrate rational functions P/Q (ratio of polynomials) by the method of **partial fractions**: If the degree of P is less than the degree of Q, then factor Q completely into linear and quadratic factors, and write P/Q as a sum of simpler terms. For example,

$$\frac{3x^2+1}{(x-1)(x+2)^2(x^2+9)} = \frac{A}{x-1} + \frac{B_1}{(x+2)} + \frac{B_2}{(x+2)^2} + \frac{Cx+D}{x^2+9}$$

Terms such as $D/(x^2+9)$ can be integrated using the trigonometric substitution $x = 3\tan u$.

This method can be used to evaluate the integral of any rational function. In practice, the hard part turns out to be factoring the denominator! In recitation you encountered two other steps required to cover every case systematically, namely, completing the square[1] and long division.[2]

3. **Integration by parts**:

$$\int_a^b uv'dx = uv\Big|_a^b - \int_a^b u'vdx$$

This is used when $u'v$ is simpler than uv'. (This is often the case if u' is simpler than u.)

4. **Arclength**: $ds = \sqrt{dx^2+dy^2}$. Depending on whether you want to integrate with respect to x, t or y this is written

$$ds = \sqrt{1+(dy/dx)^2}\,dx; \quad ds = \sqrt{(dx/dt)^2+(dy/dt)^2}\,dt; \quad ds = \sqrt{(dx/dy)^2+1}\,dy$$

5. **Surface area for a surface of revolution**:

a) around the x-axis: $2\pi y ds = 2\pi y\sqrt{1+(dy/dx)^2}\,dx$ (requires a formula for $y = y(x)$)

b) around the y-axis: $2\pi x ds = 2\pi x\sqrt{(dx/dy)^2+1}\,dy$ (requires a formula for $x = x(y)$)

6. **Polar coordinates**: $x = r\cos\theta$, $y = r\sin\theta$ (or, more rarely, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$)

a) Find the polar equation for a curve from its equation in (x, y) variables by substitution.

b) Sketch curves given in polar coordinates and understand the range of the variable θ (often in preparation for integration).

7. **Area in polar coordinates**:

$$\int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 d\theta$$

(Pay attention to the range of θ to be sure that you are not double-counting regions or missing them.)

---

[1]For example, we rewrite the denominator $x^2 + 4x + 13 = (x+2)^2 + 9 = u^2 + a^2$ with $u = x + 2$ and $a = 3$.

[2]Long division is used when the degree of P is greater than or equal to the degree of Q. It expresses $P(x)/Q(x) = P_1(x) + R(x)/Q(x)$ with P_1 a quotient polynomial (easy to integrate) and R a remainder. The key point is that the remainder R has degree less than Q, so R/Q can be split into partial fractions.

The following formulas will be printed with Exam 4

$$\sin^2 x + \cos^2 x = 1; \quad \sec^2 x = \tan^2 x + 1$$
$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x; \quad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$
$$\cos 2x = \cos^2 x - \sin^2 x; \quad \sin 2x = 2\sin x \cos x$$

$$\frac{d}{dx}\tan x = \sec^2 x; \quad \frac{d}{dx}\sec x = \sec x \tan x; \quad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}; \quad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \tan x\, dx = -\ln(\cos x) + c; \quad \int \sec x\, dx = \ln(\sec x + \tan x) + c$$

See the next page for a review on integration of rational functions.

Postscript: Systematic integration of rational functions

For a general rational function P/Q, the first step is to express P/Q as the sum of a polynomial and a ratio in which the numerator has smaller degree than the denominator.

For example,

$$\frac{x^3}{x^2-2x+1} = x+2+\frac{3x-2}{x^2-2x+1}$$

(To carry out this long division, do not factor the denominator $Q(x) = x^2 - 2x + 1$, just leave it alone.) The quotient $x + 2$ is a polynomial and is easy to integrate. The remainder term

$$\frac{3x-2}{(x-1)^2}$$

has a numerator $3x - 2$ of degree 1 which is less than the degree 2 of the denominator $(x - 1)^2$. Therefore there is a partial fraction decomposition. In fact,

$$\frac{3x-2}{(x-1)^2} = \frac{(3x-3)+1}{(x-1)^2} = \frac{3}{x-1} + \frac{1}{(x-1)^2}$$

In general, if P has degree n and Q has degree m, then long division gives

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)}$$

in which P_1, the quotient in the long division, has degree $n - m$ and R, the remainder in the long division, has degree at most $m - 1$.

Evaluation of the "simple" pieces

The integral

$$\int \frac{dx}{(x-a)^n} = \frac{-1}{n-1}(x-a)^{1-n} + c$$

if $n \neq 1$ and $\ln|x - a| + c$ if $n = 1$. On the other hand the terms

$$\int \frac{xdx}{(Ax^2+Bx+C)^n} \quad \text{and} \quad \int \frac{dx}{(Ax^2+Bx+C)^n}$$

are handled by first completing the square:

$$Ax^2 + Bx + C = A(x - B/2A)^2 + \left(C - \frac{B^2}{4A}\right)$$

Using the variable $u = \sqrt{A}(x - B/2A)$ yields combinations of integrals of the form

$$\int \frac{udu}{(u^2+k^2)^n} \quad \text{and} \quad \int \frac{du}{(u^2+k^2)^n}$$

The first integral is handled by the substitution $w = u^2 + k^2$, $dw = 2udu$. The second integral can be worked out using the trigonometric substitution $u = k \tan\theta$ $du = k \sec^2\theta d\theta$. This then leads to sec-tan integrals, and the actual computation for large values of n are long.

There are also other cases that we will not cover systematically. Examples are below:

1. If $Q(x) = (x - a)^m (x - b)^n$, then the expression is

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_n}{(x-b)^n}$$

2. If there are quadratic factors like $(Ax^2 + Bx + C)^p$, one gets terms

$$\frac{a_1x + b_1}{Ax^2 + Bx + C} + \frac{a_2x + b_2x}{(Ax^2 + Bx + C)^2} + \cdots + \frac{a_px + b_p}{(Ax^2 + Bx + C)^p}$$

for each such factor. (To integrate these quadratic pieces complete the square and make a trigonometric substitution.)

Lecture 34: Indeterminate Forms - L'Hôpital's Rule

L'Hôpital's Rule

(Two correct spellings: "L'Hôpital" and "L'Hospital")

Sometimes, we run into indeterminate forms. These are things like

$$\frac{0}{0}$$

and

$$\frac{\infty}{\infty}$$

For instance, how do you deal with the following?

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \frac{0}{0} ??$$

Example 0. One way of dealing with this is to use algebra to simplify things:

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$$

In general, when $f(a) = g(a) = 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \frac{\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

This is the easy version of L'Hôpital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Note: this only works when $g'(a) \neq 0$!

In example 0,

$$f(x) = x^3 = 1; \; g(x) = x^2 - 1$$

$$f'(x) = 3x^2; \; g'(x) = 2x \implies f'(1) = 3; \; g'(1) = 2$$

The limit is $f'(1)/g'(1) = 3/2$. Now, let's go on to the full L'Hôpital rule.

Example 1. Apply L'Hôpital's rule (a.k.a. "L'Hop") to

$$\lim_{x\to 1} \frac{x^{15}-1}{x^3-1}$$

to get

$$\lim_{x\to 1} \frac{x^{15}-1}{x^3-1} = \lim_{x\to 1} \frac{15x^{14}}{3x^2} = \frac{15}{3} = 5$$

Let's compare this with the answer we'd get if we used linear approximation techniques, instead of L'Hôpital's rule:

$$x^{15}-1 \approx 15(x-1)$$

(Here, $f(x) = x^{15}-1, a = 1, f(a) = b = 0, m = f'(1) = 15$, and $f(x) \approx m(x-a)+b$.)
Similarly,

$$x^3-1 \approx 3(x-1)$$

Therefore,

$$\frac{x^{15}-1}{x^3-1} \approx \frac{15(x-1)}{3(x-1)} = 5$$

Example 2. Apply L'Hop to

$$\lim_{x\to 0} \frac{\sin 3x}{x}$$

to get

$$\lim_{x\to 0} \frac{3\cos 3x}{1} = 3$$

This is the same as

$$\frac{d}{dx}\sin(3x)\Big|_{x=0} = 3\cos(3x)\Big|_{x=0} = 3$$

Example 3.

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x + \sin x}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$f(x) = \sin x - \cos x,\ f'(x) = \cos x + \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Remark: Derivatives $\displaystyle\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$ are always a $\displaystyle\frac{0}{0}$ type of limit.

Example 4. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x}$.
Use L'Hôpital's rule to evaluate the limit:

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{-\sin x}{x} = 0$$

Example 5. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

Just to check, let's compare that answer to the one we would get if we used quadratic approximation techniques. Remember that:

$$\cos x \approx 1 - \frac{1}{2}x^2 \quad (x \approx 0)$$

$$\frac{\cos x - 1}{x^2} \approx \frac{1 - \frac{1}{2}x^2 - 1}{x^2} = \frac{(-\frac{1}{2})x^2}{x^2} = -\frac{1}{2}$$

Example 6. $\lim_{x\to 0} \frac{\sin x}{x^2}$.

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} \quad \text{By L'Hôpital's rule}$$

If we apply L'Hôpital again, we get

$$\lim_{x\to 0} -\frac{\sin x}{2} = 0$$

But this doesn't agree with what we get from taking the linear approximation:

$$\frac{\sin x}{x^2} \approx \frac{x}{x^2} = \frac{1}{x} \to \infty \quad \text{as} \quad x \to 0^+$$

We can clear up this seeming paradox by noting that

$$\lim_{x\to 0} \frac{\cos x}{2x} = \frac{1}{0}$$

The limit is not of the form $\frac{0}{0}$, which means L'Hôpital's rule cannot be used. *The point is: look before you L'Hôp!*

More "interesting" cases that work.

It is also okay to use L'Hôpital's rule on limits of the form $\frac{\infty}{\infty}$, or if $x \to \infty$, or $x \to -\infty$. Let's apply this to rates of growth. Which function goes to ∞ faster: x, e^{ax}, or $\ln x$?

Example 7. For $a > 0$,

$$\lim_{x\to\infty} \frac{e^{ax}}{x} = \lim_{x\to\infty} \frac{ae^{ax}}{1} = +\infty$$

So e^{ax} grows faster than x (for $a > 0$).

Example 8.

$$\lim_{x\to\infty} \frac{e^{ax}}{x^{10}} = \text{by L'Hôpital} = \lim_{x\to\infty} \frac{ae^{ax}}{10x^9} = \lim_{x\to\infty} \frac{c^2 e^{ax}}{10 \cdot 9x^8} = \cdots = \lim_{x\to\infty} \frac{a^{10}e^{ax}}{10!} = \infty$$

You can apply L'Hôpital's rule ten times. There's a better way, though:

$$\left(\frac{e^{ax}}{x^{10}}\right)^{1/10} = \frac{e^{ax/10}}{x}$$

$$\lim_{x\to\infty}\frac{e^{ax}}{x^{10}} = \lim_{x\to\infty}\left(\frac{e^{ax/10}}{x}\right)^{10} = \infty^{10} = \infty$$

Example 9.

$$\lim_{x\to\infty}\frac{\ln x}{x^{1/3}} = \lim_{x\to\infty}\frac{1/x}{1/3x^{-2/3}} = \lim_{x\to\infty} 3x^{-1/3} = 0$$

Combining the preceding examples, $\ln x \ll x^{1/3} \ll x \ll x^{10} \ll e^{ax} \quad (x\to\infty, a>0)$

L'Hôpital's rule applies to $\frac{0}{0}$ and $\frac{\infty}{\infty}$. But, we sometimes face other indeterminate limits, such as 1^∞, 0^0, and $0\cdot\infty$. Use algebra, exponentials, and logarithms to put these in L'Hôpital form.

Example 10. $\lim_{x\to 0} x^x$ for $x>0$.
Because the exponent is a variable, use base e:

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x}$$

First, we need to evaluate the limit of the exponent

$$\lim_{x\to 0} x\,\ln x$$

This limit has the form $0\cdot\infty$. We want to put it in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Let's try to put it into the $\frac{0}{0}$ form:

$$\frac{x}{1/\ln x}$$

We don't know how to find $\lim_{x\to 0}\frac{1}{\ln x}$, though, so that approach isn't helpful.

Instead, let's try to put it into the $\frac{\infty}{\infty}$ form:

$$\frac{\ln x}{1/x}$$

Using L'Hôpital's rule, we find

$$\lim_{x\to 0} x\,\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0$$

Therefore,

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0}(x\,\ln x)} = e^0 = 1$$

Lecture 35: Improper Integrals

Definition.

An *improper integral*, defined by

$$\int_a^\infty f(x)dx = \lim_{M\to\infty}\int_a^M f(x)dx$$

is said to converge if the limit exists (diverges if the limit does not exist).

Example 1. $\displaystyle\int_0^\infty e^{-kx}dx = 1/k \quad (k > 0)$

$$\int_0^M e^{-kx}dx = (-1/k)e^{-kx}\Big|_0^M = (1/k)(1 - e^{-kM})$$

Taking the limit as $M \to \infty$, we find $e^{-kM} \to 0$ and

$$\int_0^\infty e^{-kx}dx = 1/k$$

We rewrite this calculation more informally as follows,

$$\int_0^\infty e^{-kx}dx = (-1/k)e^{-kx}\Big|_0^\infty = (1/k)(1 - e^{-k\infty}) = 1/k \qquad (\text{since } k > 0)$$

Note that the integral over the infinite interval $\displaystyle\int_0^\infty e^{-kx}dx = 1/k$ has an easier formula than the corresponding finite integral $\displaystyle\int_0^M e^{-kx}dx = (1/k)(1-e^{-kM})$. As a practical matter, for large M, the term e^{-kM} is negligible, so even the simpler formula $1/k$ serves as a good approximation to the finite integral. Infinite integrals are often easier than finite ones, just as infinitesimals and derivatives are easier than difference quotients.

Application: Replace x by t = time in seconds in Example 1.
R = rate of decay = number of atoms that decay per second at time 0.
At later times $t > 0$ the decay rate is Re^{-kt} (smaller by an exponential factor e^{-kt})

Eventually (over time $0 \le t < \infty$) every atom decays. So the total number of atoms N is calculated using the formula we found in Example 1,

$$N = \int_0^\infty Re^{-kt}dt = R/k$$

The half life H of a radioactive element is the time H at which the decay rate is half what it was at the start. Thus

$$e^{-kH} = 1/2 \quad\Longrightarrow\quad -kH = \ln(1/2) \quad\Longrightarrow\quad k = (\ln 2)/H$$

Hence

$$R = Nk = N(\ln 2)/H$$

Let us illustrate with Polonium 210, which has been in the news lately. The half life is 138 days or

$$H = (138\text{days})(24\text{hr/day})(60^2\text{sec/hr}) = (138)(24)(60)^2\text{seconds}$$

Using this value of H, we find that one gram of Polonium 210 emits (1 gram)($6 \times 10^{23}/210$ atoms/gram)$(ln2)/H = 1.6610^{14}$ decays/sec ≈ 4500 curies

At 5.3 MeV per decay, Polonium gives off 140 watts of radioactive energy per gram (white hot). Polonium emits alpha rays, which are blocked by skin but when ingested are 20 times more dangerous than gamma and X-rays. The lethal dose, when ingested, is about 10^{-7} grams.

Example 2. $\displaystyle\int_0^\infty dx/(1+x^2) = \pi/2$.

We calculate,

$$\int_0^M \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^M = \tan^{-1} M \to \pi/2$$

as $M \to \infty$. (If $\theta = \tan^{-1} M$ then $\theta \to \pi/2$ as $M \to \infty$. See Figures 1 and 2.)

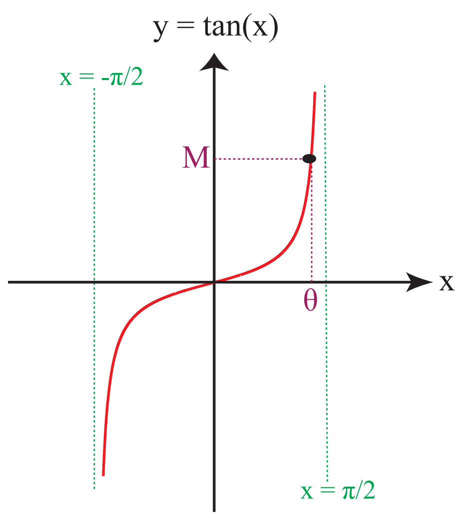

Figure 1: Graph of the tangent function, $M = \tan\theta$.

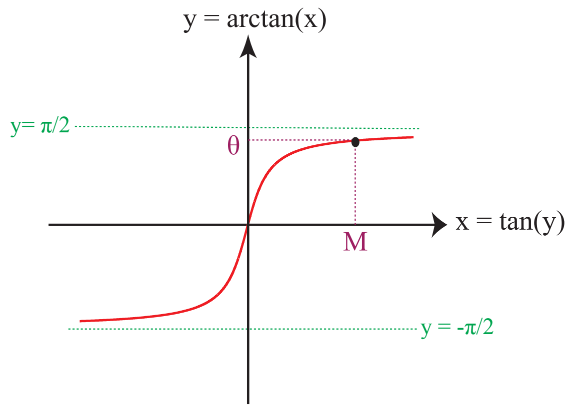

Figure 2: Graph of the arctangent function, $\theta = \tan^{-1} M$.

Example 3. $\displaystyle\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$

Recall that we already computed this improper integral (by computing a volume in two ways, slices and the method of shells). This shows vividly that a finite integral can be harder to understand than its infinite counterpart:

$$\int_0^M e^{-x^2}\,dx$$

can only evaluated numerically. It has no elementary formula. By contrast, we found an explicit formula when $M = \infty$.

Example 4. $\displaystyle\int_1^\infty dx/x$

$$\int_1^M dx/x = \ln x\Big|_1^M = \ln M - \ln 1 = \ln M \to \infty$$

as $M \to \infty$. This improper integral is infinite (called divergent or not convergent).

Example 5. $\displaystyle\int_1^\infty dx/x^p \quad (p > 1)$

$$\int_1^M dx/x^p = (1/(1-p))x^{1-p}\Big|_1^M = (1/(1-p))(M^{1-p} - 1) \to 1/(p-1)$$

as $M \to \infty$ because $1 - p < 0$. Thus, this integral is convergent.

Example 6. $\displaystyle\int_1^\infty dx/x^p \quad (0 < p < 1)$

This is very similar to the previous example, but diverges

$$\int_1^M dx/x^p = (1/(1-p))x^{1-p}\Big|_1^M = (1/(1-p))(M^{1-p} - 1) \to \infty$$

as $M \to \infty$ because $1 - p > 0$.

Determining Divergence and Convergence

To decide whether an integral converges or diverges, don't need to evaluate. Instead one can compare it to a simpler integral that can be evaluated.

The General Story for powers: $\displaystyle\int_1^\infty \frac{dx}{x^p}$
From Examples 4, 5 and 6 we know that this diverges (is infinite) for $0 < p \leq 1$ and converges (is finite) for $p > 1$.

The comparison of integrals says that a larger function has a larger integral. If we restrict ourselves to nonnegative functions, then even when the region is unbounded, as in the case of an improper integral, the area under the graph of the larger function is more than the area under the graph of the smaller one. Consider $0 \leq f(x) \leq g(x)$ (as in Figure 3)

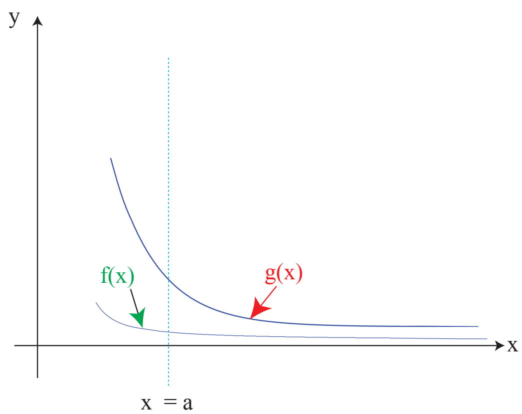

Figure 3: The area under $f(x)$ is less than the area under $g(x)$ for $a \leq x < \infty$.

If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$. (In other words, if the area under g is finite, then the area under f, being smaller, must also be finite.)

If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$. (In other words, if the area under f is infinite, then the area under g, being larger, must also be infinite.)

The way comparison is used is by replacing functions by simpler ones whose integrals we can calculate. You will have to decide whether you want to trap the function from above or below. This will depend on whether you are demonstrating that the integral is finite or infinite.

Example 7. $\int_0^\infty \frac{dx}{\sqrt{x^3+1}}$ It is natural to try the comparison

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{x^{3/2}}$$

But the area under $x^{-3/2}$ on the interval $0 < x < \infty$,

$$\int_0^\infty \frac{dx}{x^{3/2}}$$

turns out to be infinite because of the infinite behavior as $x \to 0$. We can rescue this comparison by excluding an interval near 0.

$$\int_0^\infty \frac{dx}{\sqrt{x^3+1}} = \int_0^1 \frac{dx}{\sqrt{x^3+1}} + \int_1^\infty \frac{dx}{\sqrt{x^3+1}}$$

The integral on $0 < x < 1$ is a finite integral and the second integral now works well with comparison,

$$\int_1^\infty \frac{dx}{\sqrt{x^3+1}} \leq \int_1^\infty \frac{dx}{x^{3/2}} < \infty$$

because $3/2 > 1$.

Example 8. $\int_0^\infty e^{-x^3} dx$
For $x \geq 1$, $x^3 \geq x$, so

$$\int_1^\infty e^{-x^3} dx \leq \int_1^\infty e^{-x} dx = 1 < \infty$$

Thus the full integral from $0 \leq x < \infty$ of e^{-x^3} converges as well. We can ignore the interval $0 \leq x \leq 1$ because it has finite length and e^{-x^3} does not tend to infinity there.

Limit comparison:

Suppose that $0 \leq f(x)$ and $\lim_{x\to\infty} f(x)/g(x) \leq 1$. Then $f(x) \leq 2g(x)$ for $x \geq a$ (some large a). Hence $\int_a^\infty f(x)\,dx \leq 2\int_a^\infty g(x)\,dx$.

Example 9. $\int_0^\infty \frac{(x+10)\,dx}{x^2+1}$
The limiting behavior as $x \to \infty$ is

$$\frac{(x+10)dx}{x^2+1} \simeq \frac{x}{x^2} = \frac{1}{x}$$

Since $\int_1^\infty \frac{dx}{x} = \infty$, the integral $\int_0^\infty \frac{(x+10)\,dx}{x^2+1}$ also diverges.

Example 10 (from PS8). $\int_0^\infty x^n e^{-x} dx$
This converges. To carry out a convenient comparison requires some experience with growth rates of functions.

$x^n << e^x$ not enough. Instead use $x^n/e^{x/2} \to 0$ (true by L'Hop). It follows that

$$x^n << e^{x/2} \implies x^n e^{-x} << e^{x/2}e^{-x} = e^{-x/2}$$

Now by limit comparison, since $\int_0^\infty e^{-x/2} dx$ converges, so does our integral. You will deal with this integral on the problem set.

Improper Integrals of the Second Type

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

We know that $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0$.

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 x^{-1/2} dx$$

$$\int_a^1 x^{-1/2} dx = 2x^{1/2}\Big|_a^1 = 2 - 2a^{1/2}$$

As $a \to 0$, $2a^{1/2} \to 0$. So,

$$\int_0^1 x^{-1/2}\, dx = 2$$

Similarly,

$$\int_0^1 x^{-p} dx = \frac{1}{-p+1}$$

for all $p < 1$.
For $p = \frac{1}{2}$,

$$\frac{1}{\left(-\frac{1}{2}\right)+1} = 2$$

However, for $p \geq 1$, the integral diverges.

Lecture 36: Infinite Series and Convergence Tests

Infinite Series

Geometric Series

A geometric series looks like

$$1 + a + a^2 + a^3 + ... = S$$

There's a trick to evaluate this: multiply both sides by a:

$$a + a^2 + a^3 + ... = aS$$

Subtracting,

$$(1 + a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + \cdots) = S - aS$$

In other words,

$$1 = S - aS \implies 1 = (1 - a)S \implies S = \frac{1}{1-a}$$

This only works when $|a| < 1$, i.e. $-1 < a < 1$.

$a = 1$ can't work:

$$1 + 1 + 1 + ... = \infty$$

$a = -1$ can't work, either:

$$1 - 1 + 1 - 1 + ... \neq \frac{1}{1-(-1)} = \frac{1}{2}$$

Notation

Here is some notation that's useful for dealing with series or sums. An infinite sum is written:

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + ...$$

The finite sum

$$S_n = \sum_{k=0}^{n} a_k = a_0 + ... + a_n$$

is called the "n^{th} partial sum" of the infinite series.

Definition

$$\sum_{k=0}^{\infty} a_k = s$$

means the same thing as

$$\lim_{n\to\infty} S_n = s, \text{ where } S_n = \sum_{k=0}^{n} a_k$$

We say the series *converges* to s, if the limit exists and is finite. The importance of convergence is illustrated here by the example of the geometric series. If $a = 1, S = 1 + 1 + 1 + ... = \infty$. But

$$S - aS = 1 \quad \text{or} \quad \infty - \infty = 1$$

does not make sense and is not usable!

Another type of series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

We can use integrals to decide if this type of series converges. First, turn the sum into an integral:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{dx}{x^p}$$

If that improper integral evaluates to a finite number, the series converges.

Note: This approach only tells us whether or not a series converges. It does **not** tell us what number the series converges to. That is a much harder problem. For example, it takes a lot of work to determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Mathematicians have only recently been able to determine that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges to an irrational number!

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sim \int_1^{\infty} \frac{dx}{x}$$

We can evaluate the improper integral via Riemann sums.

We'll use the upper Riemann sum (see Figure 1) to get an upper bound on the value of the integral.

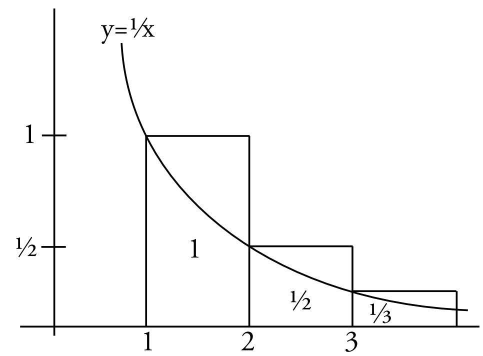

Figure 1: Upper Riemann Sum.

$$\int_1^N \frac{dx}{x} \leq 1 + \frac{1}{2} + ... + \frac{1}{N-1} = s_{N-1} \leq s_N$$

We know that

$$\int_1^N \frac{dx}{x} = \ln N$$

As $N \to \infty$, $\ln N \to \infty$, so $s_N \to \infty$ as well. In other words,

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Actually, s_N approaches ∞ rather slowly. Let's take the lower Riemann sum (see Figure 2).

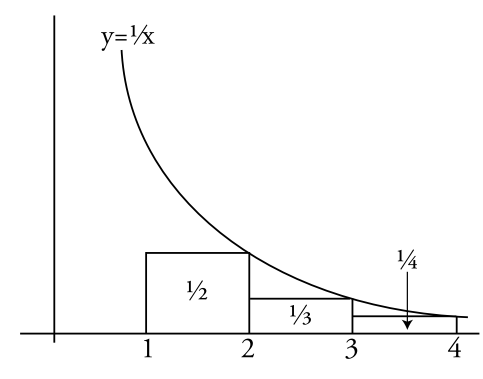

Figure 2: Lower Riemann Sum.

$$s_N = 1 + \frac{1}{2} + ... + \frac{1}{N} = 1 + \sum_{n=2}^{N} \frac{1}{n} \leq 1 + \int_1^N \frac{dx}{x} = 1 + \ln N$$

Therefore,

$$\ln N < s_N < 1 + \ln N$$

Integral Comparison

Consider a positive, decreasing function $f(x) > 0$. (For example, $f(x) = \frac{1}{x^p}$)

$$\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx \right| < f(1)$$

So, either both of the terms converge, or they both diverge. This is what we mean when we say

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{dx}{x^p}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ and converges for $p > 1$.

Lots of fudge room: in comparison.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$$

diverges, because

$$\frac{1}{\sqrt{n^2+10}} \sim \frac{1}{(n^2)^{1/2}} = \frac{1}{n}$$

Limit comparison:
If $f(x) \sim g(x)$ as $x \to \infty$, then $\sum f(n)$ and $\sum g(n)$ either both converge or both diverge.

What, exactly, does $f(x) \sim g(x)$ mean? It means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

where $0 < c < \infty$.

Let's check: does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5-10}}$$

$$\frac{n}{\sqrt{n^5-10}} \sim \frac{n}{n^{5/2}} = n^{-3/2} = \frac{1}{n^{3/2}}$$

Since $\frac{3}{2} > 1$, this series does converge.

Playing with blocks

At this point in the lecture, the professor brings out several long, identical building blocks.

Do you think it's possible to stack the blocks like this?

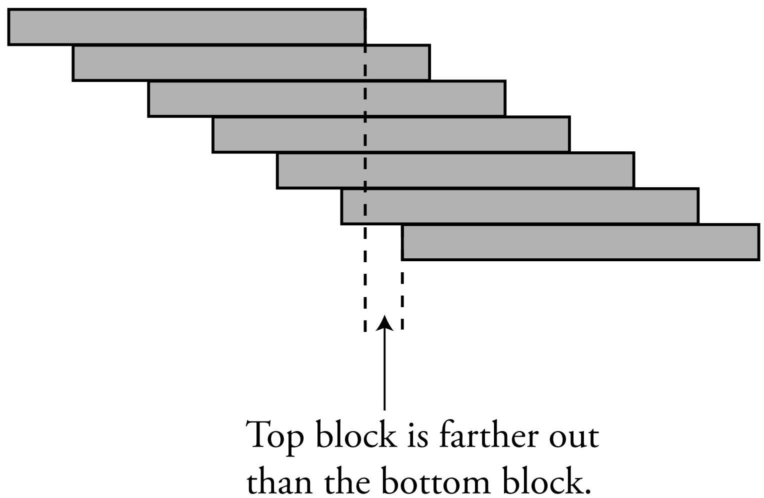

Figure 3: Collective center of mass of upper blocks is always over the base block.

In order for this to work, you want the collective center of mass of the upper blocks always to be over the base block.

The professor successfully builds the stack.

Is it possible to extend this stack clear across the room?

The best strategy is to build from the *top* block down.
Let C_0 be the left end of the first (top) block.
Let C_1 = the center of mass of the first block (top block).
Put the second block as far to the right as possible, namely, so that it's left end is at C_1 (Figure 4).
Let C_2 = the center of mass of the top two blocks.
Strategy: put the *left end* of the next block underneath the center of mass of all the previous ones combined. (See Figure 5).

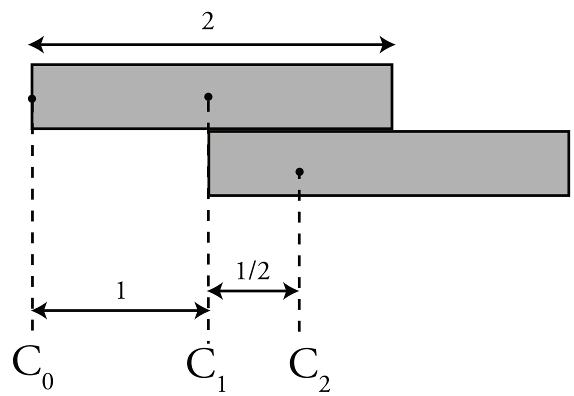

Figure 4: Stack of 2 Blocks.

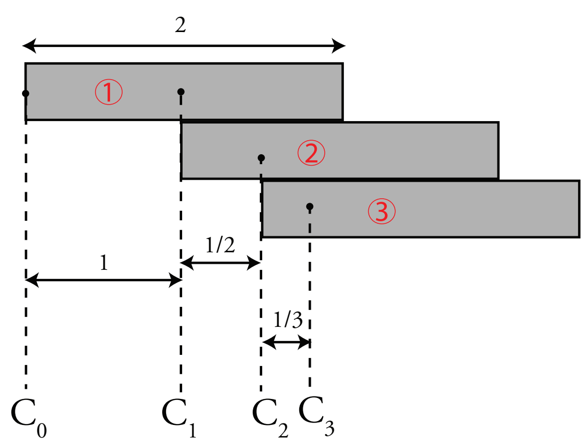

Figure 5: Stack of 3 Blocks. Left end of block 3 is C_2 = center of mass of blocks 1 and 2.

$$C_0 = 0$$

$$C_1 = 1$$

$$C_2 = 1 + \frac{1}{2}$$

$$C_{n+1} = \frac{nC_n + 1(C_n + 1)}{n+1} = \frac{(n+1)C_n + 1}{n+1} = C_n + \frac{1}{n+1}$$

$$C_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$C_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$C_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} > 2$$

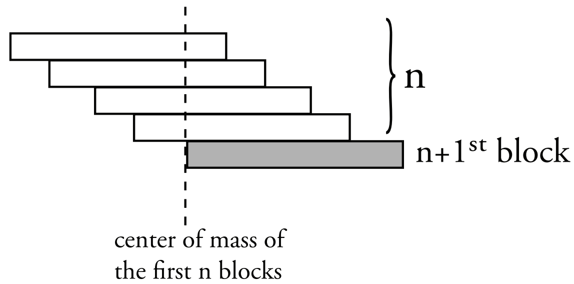

Figure 6: Stack of $n + 1$ Blocks.

So yes, you can extend this stack as far (horizontally) as you want — provided that you have enough blocks. Another way of looking at this problem is to say

$$\sum_{n=1}^{N} \frac{1}{n} = S_N$$

Recall the Riemann Sum estimation from the beginning of this lecture:

$$\ln N < S_N < (\ln N) + 1$$

as $N \to \infty$, $S_N \to \infty$.

How high would this stack of blocks be if we extended it across the two lab tables here at the front of the lecture hall? The blocks are 30 cm by 3 cm (see Figure 7). One lab table is 6.5 blocks, or 13 units, long. Two tables are 26 units long. There will be $26 - 2 = 24$ units of overhang in the stack.

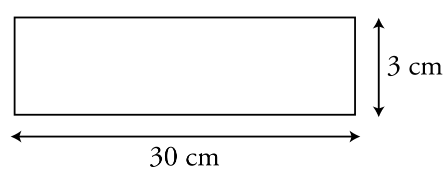

Figure 7: Side view of one block.

If $\ln N = 24$, then $N = e^{24}$.

$$\text{Height} = 3\,\text{cm} \cdot e^{24} \approx 8 \times 10^{8}\,\text{m}$$

That height is roughly twice the distance to the moon.

If you want the stack to span this room ($\sim$ 30 ft.), it would have to be 10^{26} meters high. That's about the diameter of the observable universe.

Lecture 37: Taylor Series

General Power Series

What *is* $\cos x$ anyway?

Recall: geometric series

$$1 + a + a^2 + \cdots = \frac{1}{1-a} \quad \text{for } \mid a \mid < 1$$

General power series is an infinite sum:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

represents f when $|x| < R$ where $R =$ radius of convergence. This means that for $|x| < R$, $|a_n x^n| \to 0$ as $n \to \infty$ ("geometrically"). On the other hand, if $|x| > R$, then $|a_n x^n|$ does not tend to 0. For example, in the case of the geometric series, if $|a| = \frac{1}{2}$, then $|a^n| = \frac{1}{2^n}$. Since the higher-order terms get increasingly small if $|a| < 1$, the "tail" of the series is negligible.

Example 1. If $a = -1, |a^n| = 1$ does not tend to 0.

$$1 - 1 + 1 - 1 + \cdots$$

The sum bounces back and forth between 0 and 1. Therefore it does not approach 0. Outside the interval $-1 < a < 1$, the series diverges.

Basic Tools

Rules of polynomials apply to series within the radius of convergence.

Substitution/Algebra

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Example 2. x = -u.

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$$

Example 3. $x = -v^2$.

$$\frac{1}{1+v^2} = 1 - v^2 + v^4 - v^6 + \cdots$$

Example 4.

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = (1+x+x^2+\cdots)(1+x+x^2+\cdots)$$

Term-by-term multiplication gives:

$$1+2x+3x^2+\cdots$$

Remember, here x is some number like $\frac{1}{2}$. As you take higher and higher powers of x, the result gets smaller and smaller.

Differentiation (term by term)

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[1+x+x^2+x^3+\cdots\right]$$

$$\frac{1}{(1-x)^2} = 0+1+2x+3x^2+\cdots \quad \text{where 1 is } a_0, 2 \text{ is } a_1 \text{ and 3 is } a_2$$

Same answer as Example 4, but using a new method.

Integration (term by term)

$$\int f(x)\,dx = c + \left(a_0 + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots\right)$$

where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example 5. $\displaystyle\int \frac{du}{1+u}$

$$\left(\frac{1}{1+u} = 1-u+u^2-u^3+\cdots\right)$$

$$\int \frac{du}{1+u} = c+u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\cdots$$

$$\ln(1+x) = \int_0^x \frac{du}{1+u} = x-\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}$$

So now we know the series expansion of $\ln(1+x)$.

Example 6. Integrate Example 3.

$$\frac{1}{1+v^2} = 1-v^2+v^4-v^6+\cdots$$

$$\int \frac{dv}{1+v^2} = c+\left(v-\frac{v^3}{3}+\frac{v^5}{5}-\frac{v^7}{7}+\cdots\right)$$

$$\tan^{-1} x = \int_0^x \frac{dv}{1+v^2} = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

Taylor's Series and Taylor's Formula

If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, we want to figure out what all these coefficients are. Differentiating,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$f''(x) = (2)(1)a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots$$
$$f'''(x) = (3)(2)(1)a_3 + (4)(3)(2)a_4 x + \cdots$$

Let's plug in $x = 0$ to all of these equations.

$$f(0) = a_0;\ f'(0) = a_1;\ f''(0) = 2a_2;\ f'''(0) = (3!)a_3$$

Taylor's Formula tells us what the coefficients are:

$$\boxed{f^{(n)}(0) = (n!)a_n}$$

Remember, $n! = n(n-1)(n-2)\cdots(2)(1)$ and $0! = 1$. Coefficients a_n are given by:

$$\boxed{a_n = \left(\frac{1}{n!}\right) f^{(n)}(0)}$$

Example 7. $f(x) = e^x$.

$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(n)}(x) = e^x$$
$$f^{(n)}(0) = e^0 = 1$$

Therefore, by Taylor's Formula $a_n = \dfrac{1}{n!}$ and

$$e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Or in compact form,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Now, we can calculate e to any accuracy:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Example 7. $f(x) = \cos x$.

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$f(0) = \cos(0) = 1$$
$$f'(0) = -\sin(0) = 0$$
$$f''(0) = -\cos(0) = -1$$
$$f'''(0) = \sin(0) = 0$$

Only *even* coefficients are non-zero, and their signs alternate. Therefore,

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots}$$

Note: $\cos(x)$ is an even function. So is this power series — as it contains only even powers of x.

There are two ways of finding the Taylor Series for $\sin x$. Take derivative of $\cos x$, or use Taylor's formula. We will take the derivative:

$$-\sin x = \frac{d}{dx}\cos x = 0 - 2\left(\frac{1}{2}\right)x + \frac{4}{4!}x^3 - \frac{6}{6!}x^5 + \frac{8}{8!}x^7 + \cdots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Compare with quadratic approximation from earlier in the term:

$$\boxed{\cos x \approx 1 - \frac{1}{2}x^2 \qquad \sin x \approx x}$$

We can also write:

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k = (-1)^0\frac{x^0}{0!} + (-1)^2\frac{x^2}{2!} + \cdots = 1 - \frac{1}{2}x^2 + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k \leftarrow n = 2k+1$$

Example 8: Binomial Expansion. $f(x) = (1+x)^a$

$$(1+x)^a = 1 + \frac{a}{1}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$$

Taylor Series with Another Base Point

A Taylor series with its base point at a (instead of at 0) looks like:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f^{(3)}(b)}{3!}(x-b)^3 + ...$$

Taylor series for $\sqrt{x}$. It's a bad idea to expand using $b = 0$ because $\sqrt{x}$ is not differentiable at $x = 0$. Instead use $b = 1$.

$$x^{1/2} = 1 + \frac{1}{2}(x-1) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(x-1)^2 + \cdots$$

Lecture 38: Final Review

Review: Differentiating and Integrating Series.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

Example 1: Normal (or Gaussian) Distribution.

$$\int_0^x e^{-t^2} dt = \int_0^x \left(1 - t^2 + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots\right) dt$$

$$= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - ...\right) dt$$

$$= x - \frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{5} - \frac{1}{3!}\frac{x^7}{7} + ...$$

Even though $\int_0^x e^{-t^2}\, dt$ isn't an elementary function, we can still compute it. Elementary functions are still a little bit better, though. For example:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \implies \sin\frac{\pi}{2} = \frac{\pi}{2} - \frac{(\pi/2)^3}{3!} + \frac{(\pi/2)^5}{5!} - \cdots$$

But to compute $\sin(\pi/2)$ numerically is a waste of time. We know that the sum if something very simple, namely,

$$\sin\frac{\pi}{2} = 1$$

It's not obvious from the series expansion that $\sin x$ deals with angles. Series are sometimes complicated and unintuitive.

Nevertheless, we can read this formula backwards to find a formula for $\frac{\pi}{2}$. Start with $\sin\frac{\pi}{2} = 1$. Then,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^1 = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

We want to find the series expansion for $(1-x^2)^{-1/2}$, but let's tackle a simpler case first:

$$(1+u)^{-1/2} = 1 + \left(-\frac{1}{2}\right)u + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1\cdot 2}u^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}u^3 + \cdots$$

$$= 1 - \frac{1}{2}u + \frac{1\cdot 3}{2\cdot 4}u^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}u^3 + \cdots$$

Notice the pattern: odd numbers go on the top, even numbers go on the bottom, and the signs alternate.

Now, let $u = -x^2$.

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

$$\int (1-x^2)^{-1/2}dx = C + \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}\right) + \cdots$$

$$\frac{\pi}{2} = \int_0^1 (1-x^2)^{-1/2}dx = 1 + \frac{1}{2}\left(\frac{1}{3}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1}{5}\right) + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1}{7}\right) + \cdots$$

Here's a hard (optional) extra credit problem: why does this series converge? Hint: use L'Hôpital's rule to find out how quickly the terms decrease.

The Final Exam

Here's another attempt to clarify the concept of weighted averages.

Weighted Average

A weighted average of some function, f, is defined as:

$$\text{Average}(f) = \frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}$$

Here, $\displaystyle\int_a^b w(x)\,dx$ is the total, and $w(x)$ is the weighting function.

Example: taken from a past problem set.
You get $t if a certain particle decays in t seconds. How much should you pay to play? You were given that the likelihood that the particle has not decayed (the weighting function) is:

$$w(x) = e^{-kt}$$

Remember,

$$\int_0^\infty e^{-kt}\,dt = \frac{1}{k}$$

The payoff is

$$f(t) = t$$

The expected (or average) payoff is

$$\frac{\int_0^\infty f(t)w(t)\,dt}{\int_0^\infty w(t)\,dt} = \frac{\int_0^\infty te^{-kt}\,dt}{\int_0^\infty e^{-kt}\,dt}$$

$$= k\int_0^\infty te^{-kt}\,dt = \int_0^\infty (kt)e^{-kt}\,dt$$

Do the change of variable:

$$u = kt \quad \text{and} \quad du = k\,dt$$

$$\text{Average} = \int_0^\infty ue^{-u}\,\frac{du}{k}$$

On a previous problem set, you evaluated this using integration by parts: $\displaystyle\int_0^\infty ue^{-u}\,du = 1$.

$$\text{Average} = \int_0^\infty ue^{-u}\,\frac{du}{k} = \frac{1}{k}$$

On the problem set, we calculated the half-life (H) for Polonium^{120} was $(131)(24)(60)^2$ seconds. We also found that

$$k = \frac{\ln 2}{H}$$

Therefore, the expected payoff is

$$\frac{1}{k} = \frac{H}{\ln 2}$$

where H is the half-life of the particle in seconds.

Now, you're all probably wondering: who on earth bets on particle decays?

In truth, no one does. There is, however, a very similar problem that is useful in the real world. There is something called an annuity, which is basically a retirement pension. You can buy an annuity, and then get paid a certain amount every month once you retire. Once you die, the annuity payments stop.

You (and the people paying you) naturally care about how much money you can expect to get over the course of your retirement. In this case, $f(t) = t$ represents how much money you end up with, and $w(t) = e^{-kt}$ represents how likely your are to be alive after t years.

What if you want a 2-life annuity? Then, you need multiple integrals, which you will learn about in multivariable calculus (18.02).

Our first goal in this class was to be able to differentiate anything. In multivariable calculus, you will learn about another chain rule. That chain rule will unify the (single-variable) chain rule, the product rule, the quotient rule, and implicit differentiation.

You might say the multivariable chain rule is

One thing to rule them all
One thing to find them
One thing to bring them all
And in a matrix bind them.

(with apologies to JRR Tolkien).