PROF. JERISON: We're starting a new unit today. And, so this is Unit 2, and it's called Applications of Differentiation. OK. So, the first application, and we're going to do two today, is what are known as linear approximations. Whoops, that should have two p's in it. Approximations. So, that can be summarized with one formula, but it's going to take us at least half an hour to explain how this formula is used. So here's the formula. It's f(x) is approximately equal to its value at a base point plus the derivative times x - x_0. Right? So this is the main formula. For right now. Put it in a box. And let me just describe what it means, first. And then I'll describe what it means again, and several other times.

So, first of all, what it means is that if you have a curve, which is y = f(x), it's approximately the same as its tangent line. So this other side is the equation of the tangent line. So let's give an example. I'm going to take the function f(x), which is ln x, and then its derivative is 1/x. And, so let's take the base point x_0 = 1. That's pretty much the only place where we know the logarithm for sure. And so, what we plug in here now, are the values. So f(1) is the log of 0. Or, sorry, the log of 1, which is 0. And f'(1), well, that's 1/1, which is 1. So now we have an approximation formula which, if I copy down what's right up here, it's going to be ln x is approximately, so f(0) is 0, right? Plus 1 times (x - 1). So I plugged in here, for x_0, three places. I evaluated the coefficients and this is the dependent variable.

So, all told, if you like, what I have here is that the logarithm of x is approximately x - 1. And let me draw a picture of this. So here's the graph of ln x. And then, I'll draw in the tangent line at the place that we're considering, which is x = 1. So here's the tangent line. And I've separated a little bit, but really I probably should have drawn it a little closer there, to show you. The whole point is that these two are nearby. But they're not nearby everywhere. So this is the line y = x - 1. Right, that's the tangent line. They're nearby only when x is near 1. So say in this little realm here. So when x is approximately 1, this is true. Once you get a little farther away, this straight line, this straight green line will separate from the graph. But near this place they're close together. So the idea, again, is that the curve, the curved line, is approximately the tangent line. And this is one example of it.

All right, so I want to explain this in one more way. And then we want to discuss it systematically. So the second way that I want to describe this requires me to remind you what the definition of the derivative is. So, the definition of a derivative is that it's the limit, as delta x goes to 0, of delta f / delta x, that's one way of writing it, all right? And this is the way we defined it. And one of the things that we did in the first unit was we looked at this backwards. We used the derivative knowing the derivatives of functions to evaluate some limits. So you were supposed to do that on your. In our test, there were some examples there, at least one example, where that was the easiest way to do the problem.

So in other words, you can read this equation both ways. This is really, of course, the same equation written twice. Now, what's new about what we're going to do now is that we're going to take this expression here, delta f / delta x, and we're going to say well, when delta x is fairly near 0, this expression is going to be fairly close to the limiting value. So this is approximately $f'(x_0)$. So that, I claim, is the same as what's in the box in pink that I have over here. So this approximation formula here is the same as this one. This is an average rate of change, and this is an infinitesimal rate of change. And they're nearly the same. That's the claim. So you'll have various exercises in which this approximation is the useful one to use. And I will, as I said, I'll be illustrating this a little bit today.

Now, let me just explain why those two formulas in the boxes are the same. So let's just start over here and explain that. So the smaller box is the same thing if I multiply through by delta x, as delta f is approximately $f'(x_0)$ delta x. And now if I just write out what this is, it's $f(x)$, right, minus $f(x_0)$, I'm going to write it this way. Which is approximately $f'(x_0)$, and this is $x - x_0$. So here I'm using the notations delta x is x - x0. And so this is the change in f, this is just rewriting what delta x is. And now the last step is just to put the constant on the other side. So $f(x)$ is approximately $f(x_0) + f'(x_0)(x - x_0)$. So this is exactly what I had just to begin with, right?

So these two are just algebraically the same statement. That's one another way of looking at it. All right, so now, I want to go through some systematic discussion here of several linear approximations, which you're going to be wanting to memorize. And rather than it's being hard to memorize these, it's supposed to remind you. So that you'll have a lot of extra reinforcement in remembering derivatives of all kinds. So, when we carry out these systematic discussions, we want to make things absolutely as simple as possible. And so one of the things that we do is we always use the base point to be x_0. So I'm always going to have x_0

= 0 in this standard list of formulas that I'm going to use.

And if I put x_0 = 0, then this formula becomes f(x), a little bit simpler to read. It becomes f(x) is f(0) + f'(0) x. So this is probably the form that you'll want to remember most. That's again, just the linear approximation. But one always has to remember, and this is a very important thing, this one only worked near x is 1. This approximation here really only works when x is near x_0. So that's a little addition that you need to throw in. So this one works when x is near 0. You can't expect it to be true far away. The curve can go anywhere it wants, when it's far away from the point of tangency.

So, OK, so let's work this out. Let's do it for the sine function, for the cosine function, and for e^x, to begin with. Yeah. Question.

STUDENT: [INAUDIBLE]

PROF. JERISON: Yeah. When does this one work. Well, so the question was, when does this one work. Again, this is when x is approximately x_0. Because it's actually the same as this one over here. OK. And indeed, that's what's going on when we take this limiting value. Delta x going to 0 is the same. Delta x small. So another way of saying it is, the delta x is small. Now, exactly what we mean by small will also be explained. But it is a matter to some extent of intuition as to how much, how good it is. In practical cases, people will really care about how small it is before the approximation is useful. And that's a serious issue.

All right, so let me carry out these approximations for x. Again, this is always for x near 0. So all of these are going to be for x near 0. So in order to make this computation, I have to evaluate the function. I need to plug in two numbers here. In order to get this expression. I need to know what f(0) is and I need to know what f'(0) is. If this is the function f(x), then I'm going to make a little table over to the right here with f' and then I'm going to evaluate f(0), and then I'm going to evaluate f'(0), and then read off what the answers are. Right, so first of all if the function is sine x, the derivative is cosine x. The value of f(0), that's sine of 0, is 0. The derivative is cosine. Cosine of 0 is 1. So there we go. So now we have the coefficients 0 and 1. So this number is 0. And this number is 1. So what we get here is 0 + 1x, so this is approximately x. There's the linear approximation to sin x.

Similarly, so now this is a routine matter to just read this off for this table. We'll do it for the cosine function. If you differentiate the cosine, what you get is -sin x. The value at 0 is 1, so that's cosine of 0 at 1. The value of this minus sine at 0 is 0. So this is going back over here, 1

+ 0x, so this is approximately 1. This linear function happens to be constant. And finally, if I do need e^x, its derivative is again e^x, and its value at 0 is 1, the value of the derivative at 0 is also 1. So both of the terms here, f(0) and f'(0), they're both 1 and we get 1 + x. So these are the linear approximations.

You can memorize these. You'll probably remember them either this way or that way. This collection of information here encodes the same collection of information as we have over here. For the values of the function and the values of their derivatives at 0. So let me just emphasize again the geometric point of view by drawing pictures of these results. So first of all, for the sine function, here's the sine - well, close enough. So that's - boy, now that is quite some sine, isn't it? I should try to make the two bumps be the same height, roughly speaking. Anyway the tangent line we're talking about is here. And this is y = x. And this is the function sine x. And near 0, those things coincide pretty closely. The cosine function, I'll put that underneath, I guess. I think I can fit it. Make it a little smaller here. So for the cosine function, we're up here. It's y = 1. Well, no wonder the tangent line is constant. It's horizontal. The tangent line is horizontal, so the function corresponding is constant. So this is y = cos x.

And finally, if I draw y = e^x, that's coming down like this. And the tangent line is here. And it's y = 1 + x. The value is 1 and the slope is 1. So this is how to remember it graphically if you like. This analytic picture is extremely important and will help you to deal with sines, cosines and exponentials. Yes, question.

STUDENT: [INAUDIBLE]

PROF. JERISON: The question is what do you normally use linear approximations for. Good question. We're getting there. First, we're getting a little library of them and I'll give you a few examples. OK, so now, I need to finish the catalog with two more examples which are just a little bit, slightly more challenging. And a little bit less obvious. So, the next couple that we're going to do are ln(1+x) and (1 + x)^r. OK, these are the last two that we're going to write down. And that you need to think about. Now, the procedure is the same as over here. Namely, I have to write down f' and I have to write down f'(0) and I have to write down f'(0). And then I'll have the coefficients to be able to fill in what the approximation is. So f' = 1 / (1+x), in the case of the logarithm. And f(0), if I plug in, that's log of 1, which is 0. And f' if I plug in 0 here, I get 1.

And similarly if I do it for this one, I get r(1+x)^(r-1). And when I plug in f(0), I get 1^r, which is 1. And here I get r (1)^(r-1), which is r. So the corresponding statement here is that ln(1+x) is approximately x. And (1+x)^r is approximately 1 + rx. That's 0 + 1x and here we have 1 + rx. And now, I do want to make a connection, explain to you what's going on here and the connection with the first example. We already did the logarithm once. And let's just point out that these two computations are the same, or practically the same. Here I use the base point 1, but because of my, sort of, convenient form, which will end up, I claim, being much more convenient for pretty much every purpose, we want to do these things near x is approximately 0. You cannot expand the logarithm and understand a tangent line for it at x equals 0, because it goes down to minus infinity.

Similarly, if you try to graph (1+x)^r, x^r without the 1 here, you'll discover that sometimes the slope is infinite, and so forth. So this is a bad choice of point. 1 is a much better choice of a place to expand around. And then we shift things so that it looks like it's x = 0, by shifting by the 1. So the connection with the previous example is that the-- what we wrote before I could write as ln u = u - 1. Right, that's just recopying what I have over here. Except with the letter u rather than the letter x. And then I plug in, u = 1 + x. And then that, if I copy it down, you see that I have a u in place of 1+x, that's the same as this. And if I write out u-1, if I subtract 1 from u, that means that it's x. So that's what's on the right-hand side there. So these are the same computation, I've just changed the variable.

So now I want to try to address the question that was asked about how this is used. And what the importance is. And what I'm going to do is just give you one example here. And then try to emphasize. The first way in which this is a useful idea. So, or maybe this is the second example. If you like. So we'll call this Example 2, maybe. So let's just take the logarithm of 1.1. Just a second. Let's take the logarithm of 1.1. So I claim that, according to our rules, I can glance at this and I can immediately see that it's approximately 1/10. So what did I use here? I used that ln(1+x) is approximately x, and the value of x that I used was 1/10. Right? So that is the formula, so I should put a box around these two formulas too. That's this formula here, applied with x = 1/10. And I'm claiming that 1/10 is a sufficiently small number, sufficiently close to 0, that this is an OK statement.

So the first question that I want to ask you is, which do you think is a more complicated thing. The left-hand side or the right-hand side. I claim that this is a more complicated thing, you'd have to go to a calculator to punch out and figure out what this thing is. This is easy. You know what a tenth is. So the distinction that I want to make is that this half, this part, this is hard. And this is easy. Now, that may look contradictory, but I want to just do it right above as well. This is hard. And this is easy. OK. This looks uglier, but actually this is the hard one. And this is giving us information about it.

Now, let me show you why that's true. Look down this column here. These are the hard ones, hard functions. These are the easy functions. What's easier than this? Nothing. OK. Well, yeah, 0. That's easier. Over here it gets even worse. These are the hard functions and these are the easy ones. So that's the main advantage of linear approximation is you get something much simpler to deal with. And if you've made a valid approximation you can make much progress on problems. OK, we'll be doing some more examples, but I saw some more questions before I made that point. Yeah.

STUDENT: [INAUDIBLE]

PROF. JERISON: Is this ln of 1.1 or what?

STUDENT: [INAUDIBLE]

PROF. JERISON: This is a parens there. It's ln of 1.1, it's the digital number, right. I guess I've never used that before a decimal point, have I? I don't know. Other questions.

STUDENT: [INAUDIBLE]

PROF. JERISON: OK. So let's continue here. Let me give you some more examples, where it becomes even more vivid if you like. That this approximation is giving us something a little simpler to deal with. So here's Example 3. I want to, I'll find the linear approximation near x = 0. I also - when I write this expression near x = 0, that's the same thing as this. That's the same thing as saying x is approximately 0 - of the function e^(-3x) divided by square root 1+x. So here's a function. OK. Now, what I claim I want to use for the purposes of this approximation, are just the sum of the approximation formulas that we've already derived. And just to combine them algebraically. So I'm not going to do any calculus, I'm just going to remember. So with e^(-3x), it's pretty clear that I should be using this formula for e^x. For the other one, it may be slightly less obvious but we have powers of 1+x over here. So let's plug those in. I'll put this up so that you can remember it.

And we're going to carry out this approximation. So, first of all, I'm going to write this so that it's slightly more suggestive. Namely, I'm going to write it as a product. And there you can now see the exponent. In this case, r = 1/2, eh -1/2, that we're going to use. OK. So now I have e^(-3x) (1+x)^(-1/2), and that's going to be approximately-- well I'm going to use this formula. I have to use it correctly. x is replaced by -3x, so this is 1 - 3x. And then over here, I can just copy verbatim the other approximation formula with r = -1/2. So this is times 1 - 1/2 x. And now I'm going to carry out the multiplication. So this is 1 - 3x - 1/2 x + 3/2 x^2.

So now, here's our formula. So now this isn't where things stop. And indeed, in this kind of arithmetic that I'm describing now, things are easier than they are in ordinary algebra, in arithmetic. The reason is that there's another step, which I'm now going to perform. Which is that I'm going to throw away this term here. I'm going to ignore it. In fact, I didn't even have to work it out. Because I'm going to throw it away. So the reason is that already, when I passed from this expression to this one, that is from this type of thing to this thing, I was already throwing away quadratic and higher-ordered terms. So this isn't the only quadratic term. There are tons of them. I have to ignore all of them if I'm going to ignore some of them. And in fact, I only want to be left with the linear stuff. Because that's all I'm really getting a valid computation for. So, this is approximately 1 minus, so let's see. It's a total of 7/2 x. And this is the answer. This is the linear part. So the x^2 term is negligible. So we drop x^2 term. Terms, and higher. All of those terms should be lower-order. If you imagine x is 1/10, or maybe 1/100, then these terms will end up being much smaller. So we have a rather crude approach. And that's really the simplicity, and that's the savings.

So now, since this unit is called Applications, and these are indeed applications to math, I also wanted to give you a real-life application. Or a place where linear approximations come up in real life. So maybe we'll call this Example 4. This is supposedly a real-life example. I'll try to persuade you that it is. So I like this example because it's got a lot of math, as well as physics in it. So here I am, on the surface of the earth. And here is a satellite going this way. At some velocity, v. And this satellite has a clock on it because this is a GPS satellite. And it has a time, T, OK? But I have a watch, in fact it's right here. And I have a time which I keep. Which is T', And there's an interesting relationship between T and T', which is called time dilation. And this is from special relativity. And it's the following formula. T' = T divided by the square root of 1 - v^2/c^2, where v is the velocity of the satellite, and c is the speed of light.

So now I'd like to get a rough idea of how different my watch is from the clock on the satellite. So I'm going to use this same approximation, we've already used it once. I'm going to write t. But now let me just remind you. The situation here is, we have something of the form (1-u)^(-1/2). That's what's happening when I multiply through here. So with u = v^2 / c^2. So in real life, of course, the expression that you're going to use the linear approximation on isn't necessarily itself linear. It can be any physical quantity. So in this case it's v squared over c squared. And now the approximation formula says that if this is approximately equal to, well again it's the same rule. There's an r and then x is -u, so this is - - 1/2, so it's 1 + 1/2 u. So this is approximately, by the same rule, this is T, T' is approximately t $T(1 + 1/2\ v^2/c^2)$ Now, I promised you that this would be a real-life problem. So the question is when people were designing these GPS systems, they run clocks in the satellites. You're down there, you're making your measurements, you're talking to the satellite by-- or you're receiving its signals from its radio. The question is, is this going to cause problems in the transmission. And there are dozens of such problems that you have to check for.

So in this case, what actually happened is that v is about 4 kilometers per second. That's how fast the GPS satellites actually go. In fact, they had to decide to put them at a certain altitude and they could've tweaked this if they had put them at different places. Anyway, the speed of light is $3 * 10^5$ kilometers per second. So this number, v^2 / c^2 is approximately $10^{(-10)}$. Now, if you actually keep track of how much of an error that would make in a GPS location, what you would find is maybe it's a millimeter or something like that. So in fact it doesn't matter. So that's nice. But in fact the engineers who were designing these systems actually did use this very computation. Exactly this. And the way that they used it was, they decided that because the clocks were different, when the satellite broadcasts its radio frequency, that frequency would be shifted. Would be offset. And they decided that the fidelity was so important that they would send the satellites off with this kind of, exactly this, offset. To compensate for the way the signal is. So from the point of view of good reception on your little GPS device, they changed the frequency at which the transmitter in the satellites, according to exactly this rule.

And incidentally, the reason why they didn't-- they ignored higher-order terms, the sort of quadratic terms, is that if you take u^2 that's a size $10^{(-20)}$. And that really is totally negligible. That doesn't matter to any measurement at all. That's on the order of nanometers, and it's not important for any of the uses to which GPS is put. OK, so that's a real example of a use of linear approximations.

So. let's take a little pause here. I'm going to switch gears and talk about quadratic approximations. But before I do that, let's have some more questions. Yeah.

STUDENT: [INAUDIBLE]

PROF. JERISON: OK, so the question was asked, suppose I did this by different method. Suppose I applied the original formula here. Namely, I define the function f(x), which was this function here. And then I plugged in its value at x = 0 and the value of its derivative at x = 0. So the answer is, yes, it's also true that if I call this function f f(x), then it must be true that the linear approximation is f(x_0) plus f' of - I'm sorry, it's at 0, so it's f(0), f'(0) times x. So that should be true. That's the formula that we're using. It's up there in the pink also. So this is the formula. So now, what about f(0)? Well, if I plug in 0 here, I get 1 * 1. So this thing is 1. So that's no surprise. And that's what I got. If I computed f', by the product rule it would be an annoying, somewhat long, computation. And because of what we just done, we know what it has to be. It has to be negative 7/2. Because this is a shortcut for doing it. This is faster than doing that. But of course, that's a legal way of doing it. When you get to second derivatives, you'll quickly discover that this method that I've just described is complicated, but far superior to differentiating this expression twice.

STUDENT: [INAUDIBLE] PROF. JERISON: Would you have to throw away an x^2 term if you differentiated? No. And in fact, we didn't really have to do that here. If you differentiate and then plug in x = 0. So if you differentiate this and you plug in x = 0, you get -7/2. You differentiate this and you plug in x = 0, this term still drops out because it's just a 3x when you differentiate. And then you plug in x = 0, it's gone too. And similarly, if you're up here, it goes away and similarly over here it goes away. So the higher-order terms never influence this computation here. This just captures the linear features of the function.

So now I want to go on to quadratic approximation. And now we're going to elaborate on this formula. So, linear approximation. Well, that should have been linear approximation. Liner. That's interesting. OK, so that was wrong. But now we're going to change it to quadratic. So, suppose we talk about a quadratic approximation here. Now, the quadratic approximation is going to be just an elaboration, one more step of detail. From the linear. In other words, it's an extension of the linear approximation. And so we're adding one more term here. And the extra term turns out to be related to the second derivative. But there's a factor of 2. So this is the formula for the quadratic approximation. And this chunk of it, of course, is the linear part. This time I'll spell 'linear' correctly. So the linear part is the first piece. And the quadratic part is the second piece.

I want to develop this same catalog of functions as I had before. In other words, I want to extend our formulas to the higher-order terms. And if you do that for this example here, maybe

I'll even illustrate with this example before I go on, if you do it with this example here, just to give you a flavor for what goes on, what turns out to be the case. So this is the linear version. And now I'm going to compare it to the quadratic version. So the quadratic version turns out to be this. That's what turns out to be the quadratic approximation. And when I use this example here, so this is 1.1, which is the same as ln of 1 + 1/10, right? So that's approximately 1/10 - 1/2 $(1/10)^2$. So 1/200. So that turns out, instead of being 1/10, that's point, what is it, .095 or something like that. It's a little bit less. It's not .1, but it's pretty close. So if you like, the correction is lower in the decimal expansion.

Now let me actually check a few of these. I'll carry them out. And what I'm going to probably save for next time is explaining to you, so this is why this factor of 1/2, and we're going to do this later. Do this next time. You can certainly do well to stick with this presentation for one more lecture. So we can see this reinforced. So now I'm going to work out these derivatives of the higher-order terms. And let me do it for the x approximately 0 case. So first of all, I want to add in the extra term here. Here's the extra term. For the quadratic part. And now in order to figure out what's going on, I'm going to need to compute, also, second derivatives. So here I need a second derivative. And I need to throw in the value of that second derivative at 0. So this is what I'm going to need to compute. So if I do it, for example, for the sine function, I already have the linear part. I need this last bit. So I differentiate the sine function twice and I get, I claim minus the sine function. The first derivative is the cosine and the cosine derivative is minus the sine.

And when I evaluate it at 0, I get, lo and behold, 0. Sine of 0 is 0. So actually the quadratic approximation is the same. $0x^2$. There's no x^2 term here. So that's why this is such a terrific approximation. It's also the quadratic approximation. For the cosine function, if you differentiate twice, you get the derivative is minus the sign and derivative of that is minus the cosine. So that's f''. And now, if I evaluate that at 0, I get -1. And so the term that I have to plug in here, this -1 is the coefficient that appears right here. So I need a -1/2 x^2 extra. And if you do it for the e^x, you get an e^x, and you got a 1 and so you get 1/2 x^2 here.

I'm going to finish these two in just a second, but I first want to tell you about the geometric significance of this quadratic term. So here we go. Geometric significance (of the quadratic term). So the geometric significance is best to describe just by drawing a picture here. And I'm going to draw the picture of the cosine function. And remember we already had the tangent line. So the tangent line was this horizontal here. And that was y = 1. But you can see intuitively, that doesn't even tell you whether this function is above or below 1 there. Doesn't tell you much. It's sort of begging for there to be a little more information to tell us what the function is doing nearby. And indeed, that's what this second expression does for us. It's some kind of parabola underneath here. So this is $y = 1 - 1/2 x^2$. Which is a much better fit to the curve than the horizontal line. And this is, if you like, this is the best fit parabola. So it's going to be the closest parabola to the curve. And that's more or less the significance. It's much, much closer.

All right, I want to give you, well, I think we'll save these other derivations for next time because I think we're out of time now. So we'll do these next time.