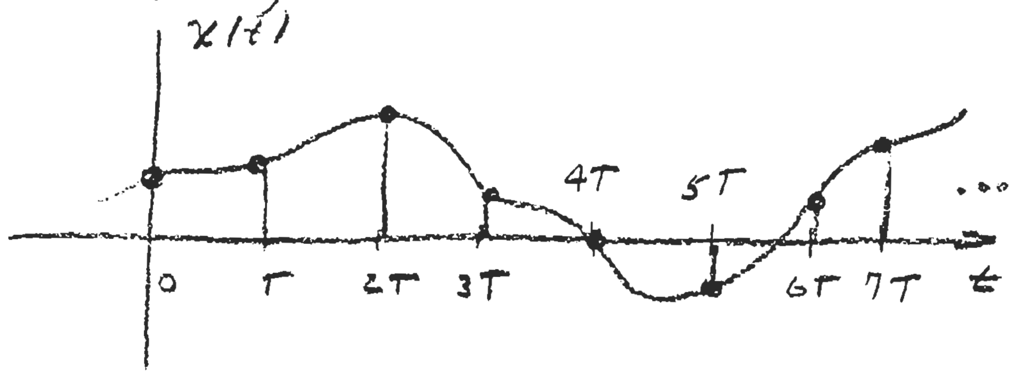
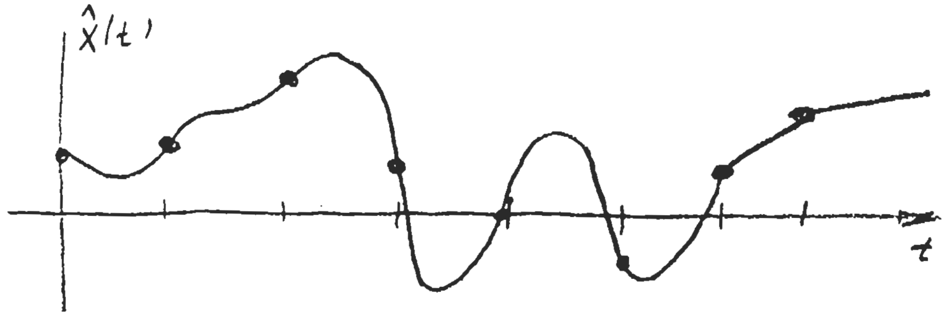
LECTURE S24

Sampling

Especially when working with computers, it is useful to represent a signal by sampling it:



Problem — the samples do not uniquely determine the signal:

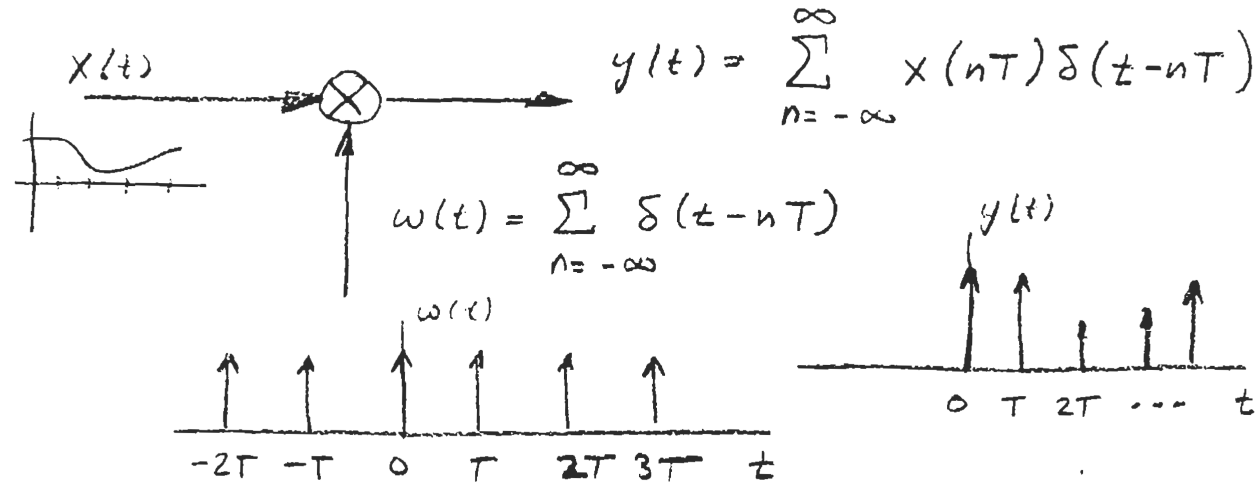


What is the right way to reconstruct the original signal?

How rapidly should we sample?

Modelling the Sampling Process

We can model the sampling process as a kind of modulation:



$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$$w(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

The FT of $y(t)$ is

$$Y(f) = X(f) * W(f)$$

What is $W(f) = \mathcal{F}[w(t)]$? Answer is a little tricky.

~~Because~~ $w(t)$ is <u>periodic</u>, can represent by a Fourier Series:

$$w(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2n\pi t/T}$$

where $$a_n = \frac{1}{T}\int_{-T/2}^{T/2} w(t)\, e^{-j2n\pi t/T}\, dt$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-j2n\pi t/T}\, dt$$

$$= \frac{1}{T}, \text{ for all } n.$$

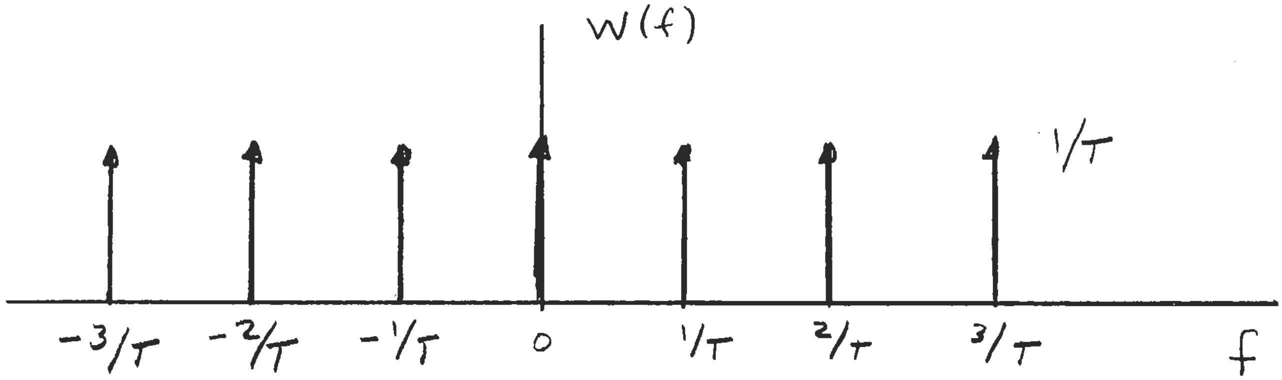
$$\therefore,\ w(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2n\pi t/T}$$

Note: The sum above doesn't converge
Let's ignore this.

Then $$W(f) = \mathcal{F}[w(t)]$$

$$= \mathcal{F}\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T}\right]$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f - n/T)$$



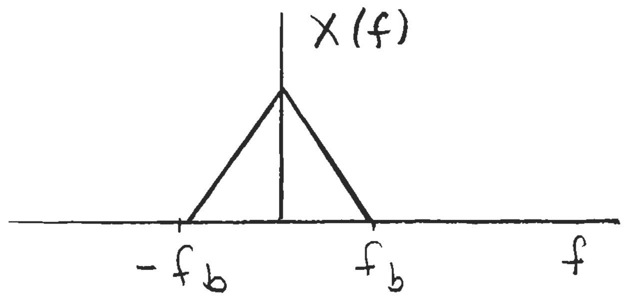
The FT of a periodic pulse train is a periodic pulse train!

So,

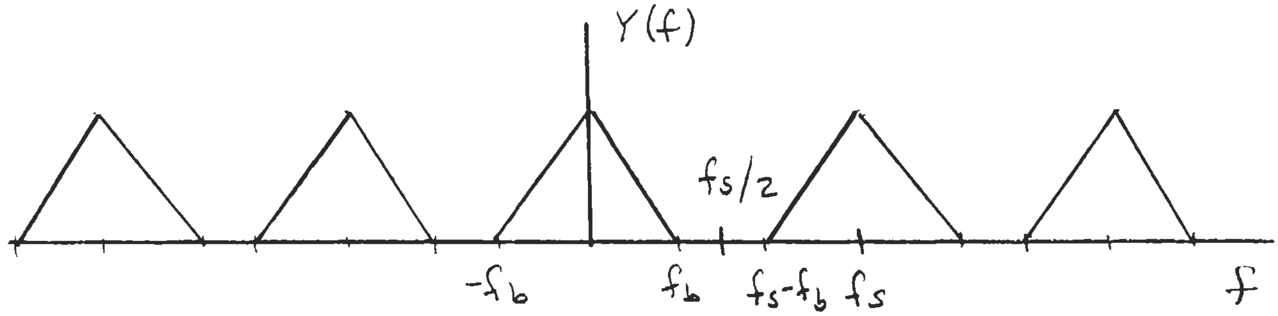
$$Y(f) = X(f) * W(f)$$

$$= X(f) * \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - n/T)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X(f - n/T)$$



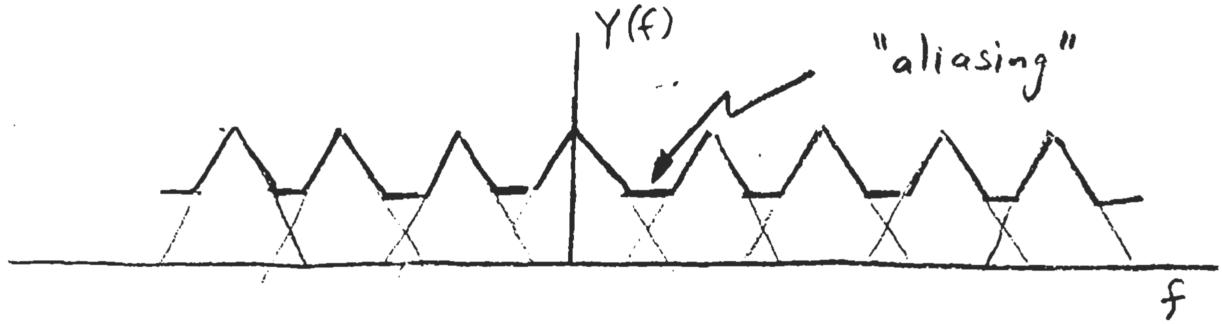
f_b = "bandwidth" of $X(f)$



$f_s = 1/T$

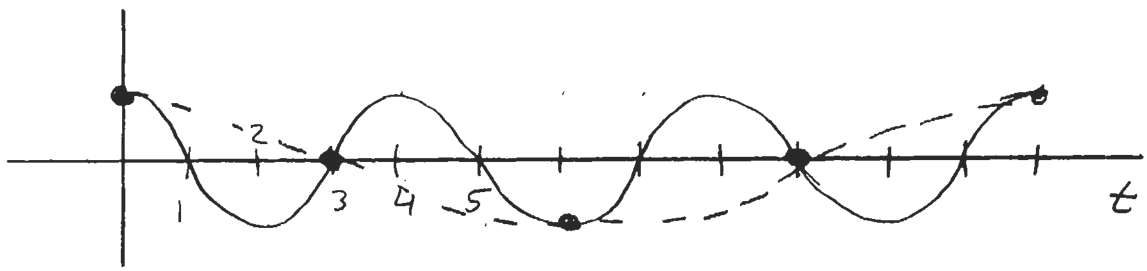
So if $f_s > 2f_b$, we have (many!) exact copies of $X(f)$.

What happens if $f_s < 2f_b$?



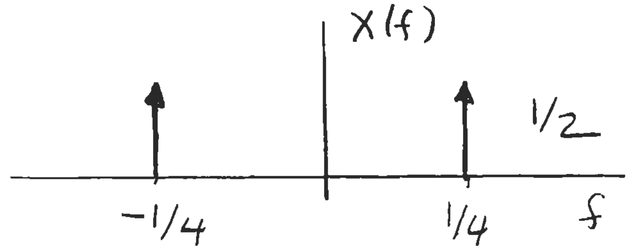
Spectrum of $X(f)$ is no longer intact, so information is lost.

<u>Example</u>



$x(t) = \cos \frac{\pi t}{2}$ $\qquad X(f) = \frac{1}{2}\Big(\delta(f - 1/4) + \delta(f + 1/4)\Big)$

$f_b = 1/4 \quad \Longleftarrow$



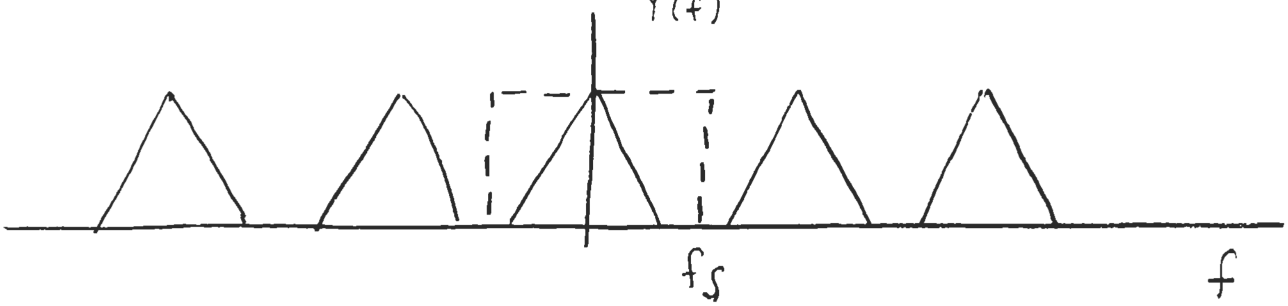
If $T = 3$, $f_s = 1/3 < 2 \cdot f_b$. So <u>aliasing</u> occurs. Samples signal looks like it comes from

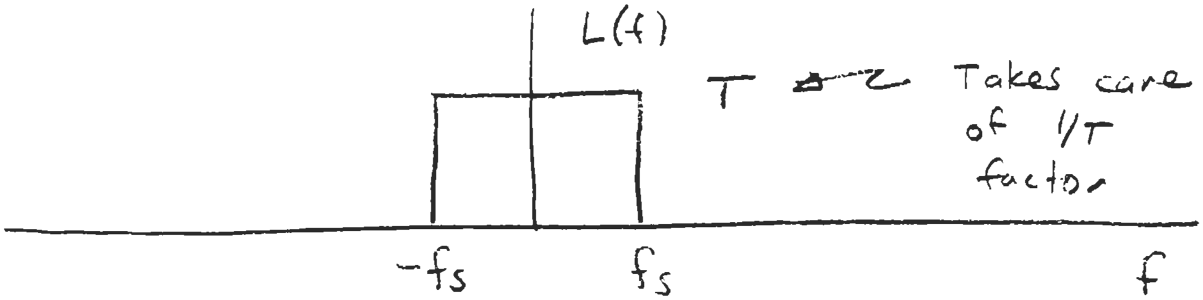
$$\cos \frac{\pi}{6} t = \text{"alias" of } \cos \frac{\pi t}{2}$$

$$(f = 1/12) \qquad\qquad (f = 1/4)$$

So assume $f_s > 2 f_b$ = "Nyquist frequency"

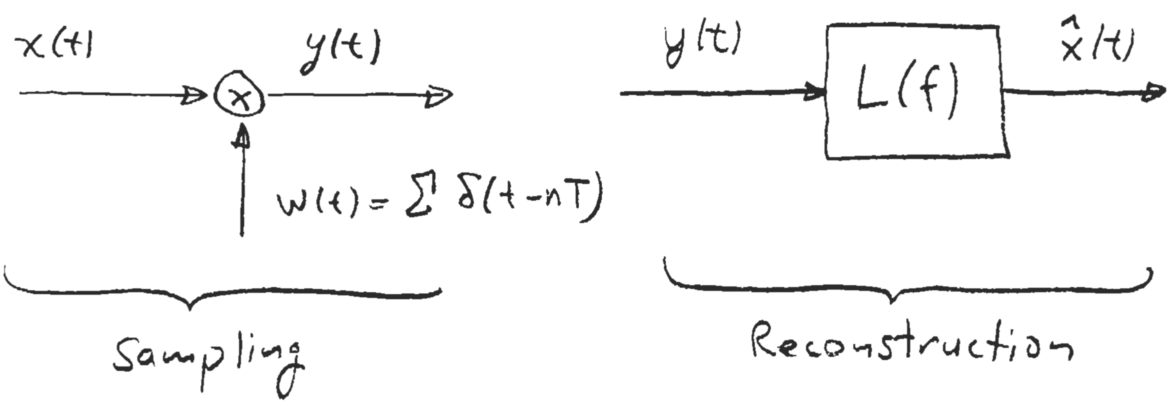
How can we reconstruct the signal?
Use a low-pass filter to eliminate repeated spectra.





$$L(f) = \begin{cases} T, & |f| \le f_s/2 \\ 0, & |f| \geq f_s/2 \end{cases}$$

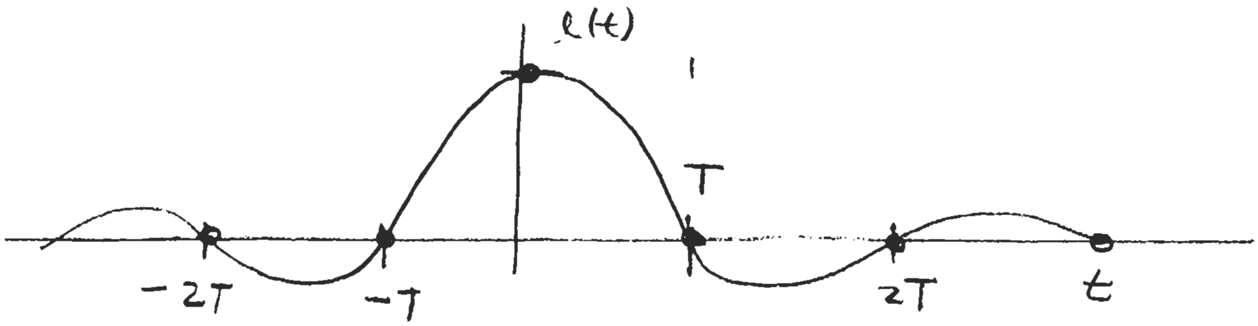
So we have:



Impulse response of filter is

$$\ell(t) = \mathcal{F}^{-1}[L(f)]$$

$$= \frac{\sin \pi t/T}{\pi t/T} = \operatorname{sinc} \frac{\pi t}{T}$$



So reconstructed signal is

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT) \operatorname{sinc}\left(\frac{\pi}{T}(t-nT)\right)$$