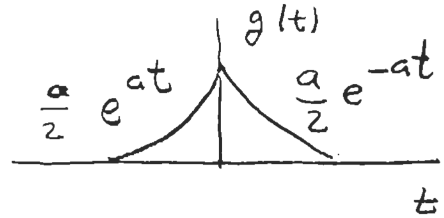
LECTURE S15

Examples of Bilateral Laplace Transforms

Smoother:



$$g(t) = \frac{a}{2} e^{at} \sigma(-t) + \frac{a}{2} e^{-at} \sigma(t)$$

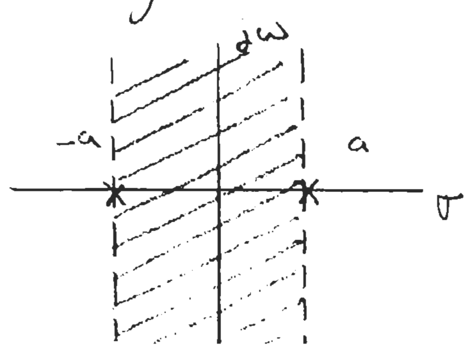
$$\mathcal{L}[g(t)] = \frac{a}{2} \frac{-1}{s-a} + \frac{a}{2} \frac{1}{s+a}$$

$$\mathrm{Re}[s] < a \qquad\qquad \mathrm{Re}[s] > -a$$

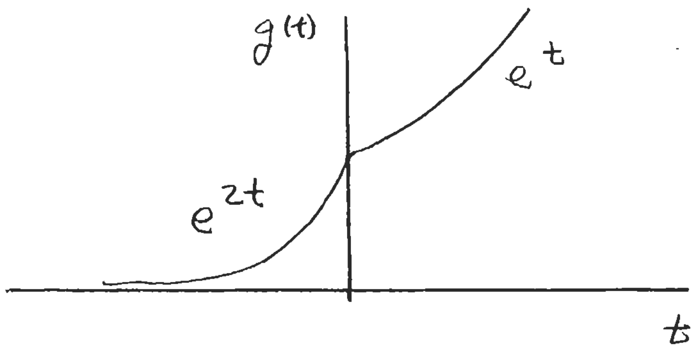
$$= \frac{a}{2} \frac{-s-a+s-a}{(s-a)(s+a)}$$

$$= \frac{-a^2}{s^2-a^2}, \quad -a < \mathrm{Re}[s] < a$$

Region of convergence:



Example $g(t) = e^{2t}\sigma(-t) + e^{+t}\sigma(t)$

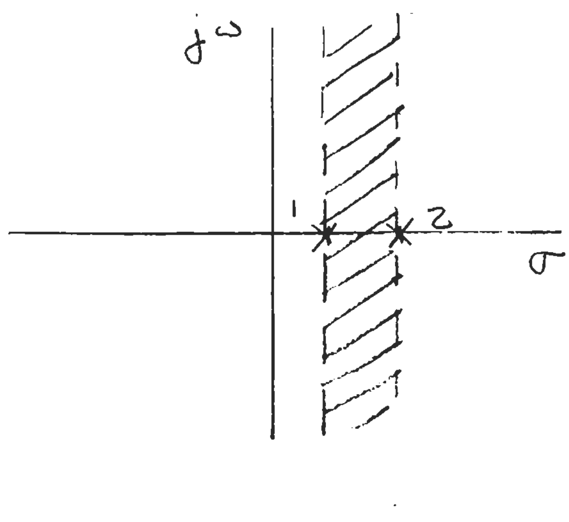


$$G(s) = \frac{-1}{s-2} + \frac{1}{s-1}$$

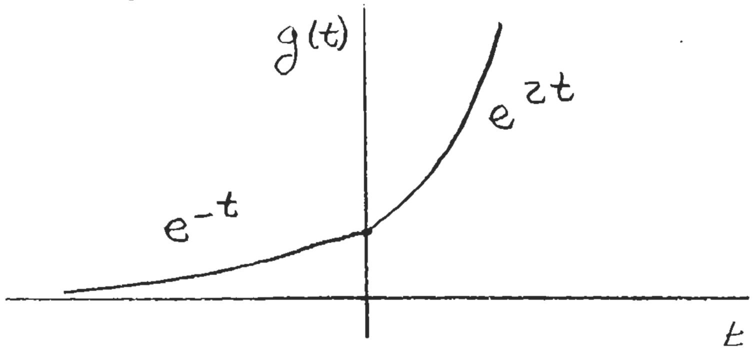
$$Re[s] < 2 \qquad Re[s] > 1$$

$$= \frac{-1}{(s-1)(s-2)}, \qquad 1 < Re[s] < 2$$

Region of convergence:



Example: $g(t) = e^{t}\sigma(-t) + e^{2t}\sigma(t)$



$$G(s) = \frac{-1}{s-1} + \frac{1}{s-2}$$

$$Re[s] < 1 \qquad\qquad Re[s] > 2$$

There is no region of convergence!
So there is no Laplace transform.

<u>Summary</u>:

1) The LT is easily extended to noncausal signals
2) Two different signals can have the same functional form for their LTs, but with different regions of convergence.
3) Not every signal has a LT!

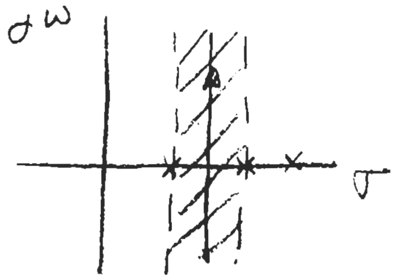
The Inverse Laplace Transform

There is a formula for the inverse Laplace transform:

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} G(s) e^{st} ds \qquad (*)$$

where c is in the region of convergence



Notes:

1) <u>Proof</u> of this is difficult. Basic idea is to express $G(s) = \int g(t) e^{-st} dt$. Then right side of $(*)$ is a double integral. The inner integral only makes sense in the r.o.c. So for the integration path chosen, can reverse order of integration. Evaluating the integral gives $g(t)$.
(see next page)

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} G(s)\,e^{st}\,ds$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\left\{\int_{-\infty}^{\infty} g(\tau)\,e^{-s\tau}\,d\tau\right\} e^{st}\,ds$$

$$= \int_{-\infty}^{\infty} g(\tau)\underbrace{\int_{c-j\infty}^{c+j\infty}\frac{e^{-s\tau}e^{st}}{2\pi j}\,ds}_{\delta(t-\tau)}\,d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau)\,\delta(t-\tau)\,d\tau = g(t)$$

The only suspect part of the proof is the statement that

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{-s\tau}e^{st}\,ds = \delta(t-\tau)$$

This can be shown using a limiting process. Strictly speaking, the sum does not converge.

Notes

2) To make formula useful, need calculus of complex variables (18.075)....

3) ... but this reduces to partial fraction expansion!

Example $G(s) = \dfrac{s+2}{(s+1)(s+3)}$, $-3 < s < -1$

what is $g(t)$?

$$G(s) = \frac{1/2}{s+1} + \frac{1/2}{s+3}$$

$$(s < -1) \qquad (s > -3)$$

$$\Rightarrow g(t) = \frac{-1}{2} e^{-t} \sigma(-t) + \frac{1}{2} e^{-3t} \sigma(t)$$

Note negative sign!

Answer would be different for different r.o.c.