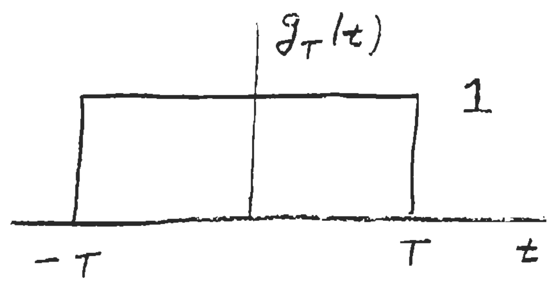
LECTURE ∶18

The FT of Special Functions

What is the FT of $g(t) = 1$? $g(t)$ is not stable, but we can find the result in the limit. Take

$$g_T(t) = \begin{cases} 1, & |t| \le T \\ 0, & |t| > T \end{cases}$$



Then

$$\lim_{T \to \infty} g_T(t) = g(t) = 1$$

So

$$\mathcal{F}[g(t)] = \lim_{T \to \infty} \mathcal{F}[g_T(t)]$$

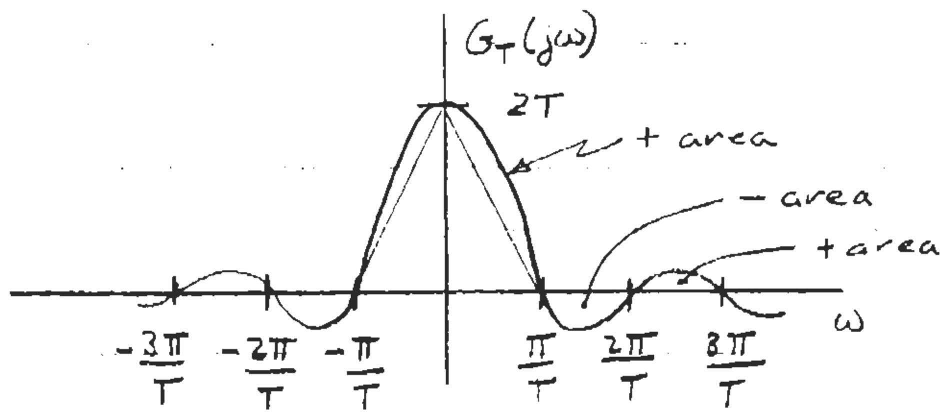
So let's do the math.

$$\mathcal{F}[g_T(t)] = \int_{-T}^{T} e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega} \, e^{-j\omega t} \Big|_{-T}^{T}$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega T} - e^{+j\omega T}\right] = 2\,\frac{\sin \omega T}{\omega}$$

$$= 2T\,\frac{\sin \omega T}{\omega T} = 2T \operatorname{sinc} \omega T$$



As $T \to \infty$, height goes to ∞, width goes to 0, but area is constant! So looks like a impulse.

Can <u>estimate</u> area as

$$\text{Area} \approx \frac{1}{2}\left(\frac{2\pi}{T}\right) \cdot 2T = 2\pi$$

(area of triangle)

To find area exactly, use the following trick:

$$\int_{-\infty}^{\infty} G_T(j\omega)\, d\omega = \left. \int_{-\infty}^{\infty} G_T(j\omega)\, e^{j\omega t}\, d\omega \right|_{t=0}$$

$$= 2\pi\, g_T(t) \Big|_{t=0} = 2\pi\, g_T(0)$$

$$= 2\pi$$

So Area $= 2\pi$, as in estimate.

Conclusion:

$$\mathcal{F}[1] = 2\pi\delta(\omega) = \delta(f)$$

Do inverse FT to check:

$$\mathcal{F}^{-1}[2\pi\,\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)\, e^{j\omega t}\, d\omega$$

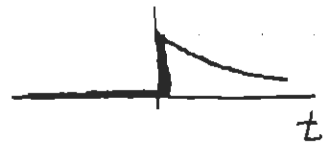
$$= e^{j\omega t}\Big|_{\omega=0} \qquad \text{(sifting property)}$$

$$= 1, \quad \text{all } t. \quad \checkmark$$

<u>Problem</u> What is $\mathcal{F}[\sigma(t)]$?

Again, use a limiting process. Let

$$g_a(t) = \sigma(t)\, e^{-at}$$



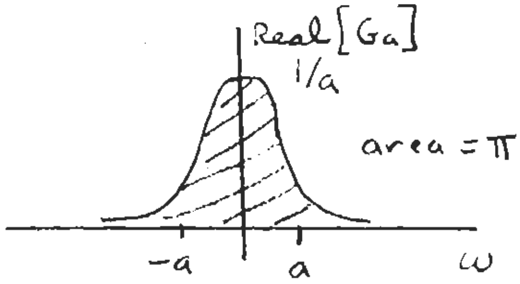
and take limit $a \to 0$.

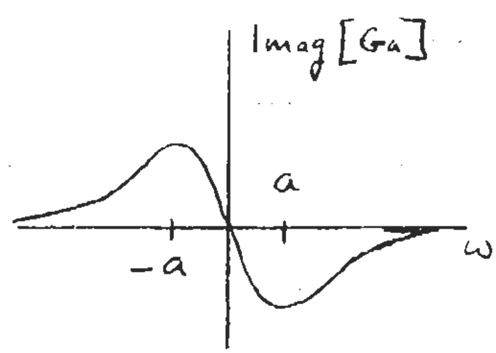
$$\mathcal{F}[g_a(t)] = \frac{1}{j\omega + a} \equiv G_a(j\omega)$$

As $a \to 0$, $G_a(j\omega) \to \frac{1}{j\omega}$, point by point.
<u>But</u>, must be careful when taking limits of functions.

$$G_a(j\omega) = \frac{1}{j\omega + a}\,\frac{-j\omega + a}{-j\omega + a}$$

$$= \underbrace{\frac{a}{\omega^2 + a^2}}_{\text{real part}} + j\,\underbrace{\frac{(-\omega)}{a^2 + \omega^2}}_{\text{imaginary part}}$$





$$\lim_{a \to 0} G_a(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

New to FT

expected from LT

Convolution and the Fourier Transform

From our experience with Laplace transforms, we know that

$$\mathcal{F}[g(t) * u(t)] = \mathcal{F}[g(t)]\,\mathcal{F}[u(t)]$$

$$= G(f)U(f)$$

$$(\text{or } G(j\omega)U(j\omega))$$

Instead of convolution in the time domain, what if we do multiplication. That is, what is

$$\mathcal{F}[g(t)u(t)] \ ?$$

Use time-frequency duality! Or, evaluate directly.

Result is

$$\mathcal{F}[g(t)u(t)] = G(f) * U(f)$$

$$\left(\text{or } \frac{1}{2\pi} G(j\omega) * U(j\omega)\right)$$

Proof:

Define $\quad H(j\omega) = G(j\omega) * U(j\omega)$

convolution w.r.t. ω

Then

$$h(t) = \mathcal{F}^{-1}[H(j\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\int_{-\infty}^{\infty} G(j(\omega - v))U(jv)\,dv\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\omega-v)t}\,e^{jvt}\,G(j(\omega-v))U(jv)\,dv\,d\omega$$

Let $\omega' = \omega - v$. Then

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega' t}e^{jvt}\,G(j\omega')\,U(jv)\,dv\,d\omega'$$

$$h(t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{j\omega' t}\, G(j\omega')\, d\omega'\right) \cdot$$

$$\left(\int_{-\infty}^{\infty} e^{j\nu t}\, U(j\nu)\, d\nu\right)$$

$$= 2\pi\, g(t)u(t)$$

Therefore,

$$g(t)\,u(t) = \frac{1}{2\pi}\,\mathcal{F}^{-1}\left[G(j\omega) * U(j\omega)\right]$$

$$\Rightarrow \mathcal{F}\left[g(t)\,u(t)\right] = \frac{1}{2\pi}\,G(j\omega) * U(j\omega)$$

$$= G(f) * U(f)$$