A system has step response given by

$$g_s(t) = \begin{cases} 0, & t < 0 \\ e^{-t} + e^{-3t}, & t \geq 0 \end{cases} = \left(e^{-t} + e^{-3t}\right)\sigma(t)$$

Find and plot the response of the system to the input

$$u(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-2t}, & t \geq 0 \end{cases} = \left(1 - e^{-2t}\right)\sigma(t)$$

using Duhamel's integral.

Duhamel's integral is

$$y(t) = u(0)\,g_s(t) + \int_0^\infty g_s(t-\tau)\,u'(\tau)\,d\tau$$

In this case, $u(0) = 0$, so

$$y(t) = \int_0^\infty g_s(t-\tau)\,u'(\tau)\,d\tau$$

$$= \int_0^\infty \left[e^{-(t-\tau)} + e^{-3(t-\tau)}\right]\sigma(t-\tau) \cdot 2e^{-2\tau}\,d\tau$$

Note that

$$\sigma(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$$

Therefore,

$$y(t) = \int_0^t \left[e^{-(t-\tau)} + e^{-3(t-\tau)}\right] 2e^{-2\tau}\,d\tau$$

$$= \int_0^t 2e^{-t}e^{-\tau} + 2e^{-3t}e^{\tau}\,d\tau$$

$$= 2e^{-t}\frac{e^{-\tau}}{-1} + 2e^{-3t}\frac{e^{\tau}}{1}\Bigg|_{\tau=0}^{t}$$

Therefore,

$$y(t) = -2e^{-t}\left[e^{-t}-1\right] + 2e^{-3t}\left[e^{t}-1\right]$$

$$= -2e^{-2t} + 2e^{-t} + 2e^{-2t} - 2e^{-3t}$$

$$= 2e^{-t} - 2e^{-3t}, \qquad t \geq 0$$

$$= 0, \qquad t < 0$$

$$\boxed{y(t) = \left[2e^{-t} - 2e^{-3t}\right]\sigma(t)}$$