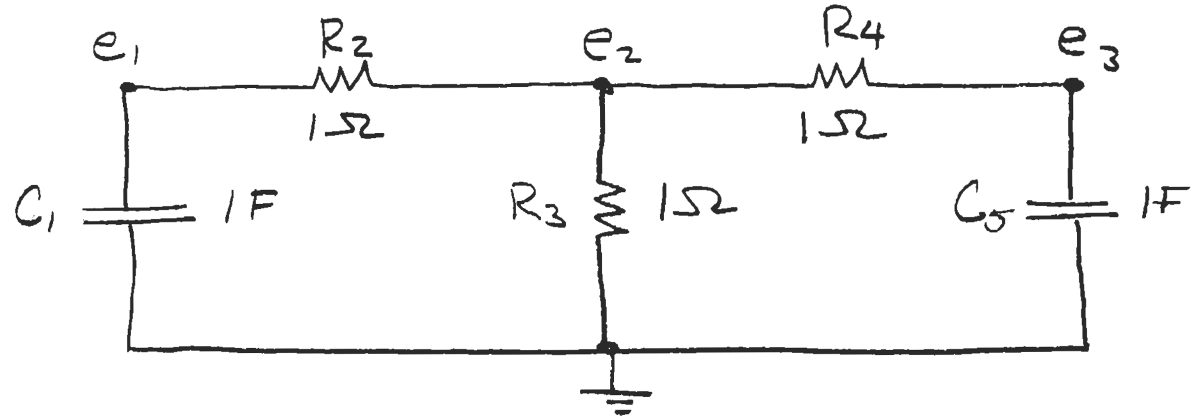
Lecture 8

Time Response of RC Networks

Example:



Given initial conditions $v_1(0)$, $v_5(0)$,
How do we find time histories of circuit variables (v's, i's)?

First, find differential equations using node method

$$* \; e_1: \quad C_1 \dot{e}_1 + G_2 e_1 \qquad - G_2 e_2 \qquad = 0$$

$$** \; e_2: \quad - G_2 e_1 + (G_2 + G_3 + G_4) e_2 \quad - G_4 e_3 = 0$$

$$* \; e_3: \quad - G_4 e_2 + C_5 \dot{e}_3 + G_4 e_3 = 0$$

$*$ Dynamic – It's a differential equation for e_1 or e_3 in terms of e_1, e_2, e_3

$**$ Static - Gives e_2 in terms of e_1, e_3

How do we solve the differential equations? Guess! (But an educated guess.)

Our guess:

$$e_1(t) = E_1 e^{st}$$
$$e_2(t) = E_2 e^{st}$$
$$e_3(t) = E_3 e^{st}$$

E_1, E_2, E_3 are <u>amplitudes</u>.

s is the same number for each exponential, that is, there is <u>not</u> s_1, s_2, s_3.

Is this a good guess? Try solution and see!

$$e_1: \quad C_1 \dot{e}_1(t) + G_2 e_1(t) - G_2 e_2(t)$$

$$= C_1 \frac{d}{dt}\left(E_1 e^{st}\right) + G_2\left(E_1 e^{st}\right) - G_2\left(E_2 e^{st}\right)$$

$$= C_1 E_1 s e^{st} + G_2 E_1 e^{st} - G_2 E_2 e^{st} = 0$$

$$\Rightarrow \quad C_1 s E_1 + G_2 E_1 - G_2 E_2 = 0$$

$$\Rightarrow \quad (C_1 s + G_2) E_1 - G_2 E_2 = 0$$

Note that $C_1 s$ + G_2 have same units.

G_2 = conductance of resistor R_2

$\Rightarrow C_1 s$ = "~~conductance~~" of capacitor C_1

↳ experts say "admittance"

$\Rightarrow \dfrac{1}{C_1 s}$ = "~~resistance~~" of capacitor C_1

↳ experts say "impedance"

Can do the same for other equations:

$$(C_1 s + G_2) E_1 - G_2 E_2 = 0$$

$$-G_2 E_1 + (G_2 + G_3 + G_4) E_2 - G_4 E_3 = 0$$

$$-G_4 E_2 + (C_5 s + G_4) E_3 = 0$$

In matrix form,

$$\begin{bmatrix} C_1 s + G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 + G_4 & -G_4 \\ 0 & -G_4 & C_5 s + G_4 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or, $M(s)\,\underline{E} = \underline{0}$ vectors

One obvious solution is $\underline{E} = \underline{0}$.
But this is a <u>trivial</u> solution — it doesn't help at all.

There is a <u>non-trivial</u> solution only if

$$\phi(s) = \det(M(s)) = 0$$ "characteristic equation"

At this point, plug in numbers:

$$M(s) = \begin{bmatrix} s+1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & s+1 \end{bmatrix}$$

(see linear algebra handout to review how to take determinant)

$$\phi(s) = |M(s)| = 3(s+1)(s+1) - (-1)(-1)(s+1) - (-1)(-1)(s+1)$$

$$= 3s^2 + 4s + 1$$

$$= (3s+1)(s+1) = 0$$

Solutions are

$s = -1 \text{ sec}^{-1} = s_1$ or $s = -1/3 \text{ sec}^{-1} = s_2$

These are the "characteristic values"

Now, solve for E_1, E_2, E_3

<u>$s = s_1 = -1$:</u>

$$M(-1)\underline{\underline{E}}' = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \underline{\underline{E}}' = \underline{0}$$

Do row reductions (see linear alg. primer!)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{\underline{E}}' = \underline{0}$$

A solution is

$$\underline{\underline{E}}' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ (or any multiple)

"characteristic vector"

So one solution is

$$\left.\begin{aligned} e_1 &= -1e^{-t} \\ e_2 &= 0 \\ e_2 &= 1e^{-t} \end{aligned}\right\} \text{(or any multiple)}$$

$\underline{s = s_2 = -1/3 \text{ sec}^{-1}}$:

$$M(-1/3) = \begin{bmatrix} 2/3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2/3 \end{bmatrix}$$

Do row reductions:

$$\begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \underline{E}^2 = \underline{0}$$

One solution is

$$\underline{E}^2 = \begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}$$

So I have two possible solutions:

$$\underline{e}(t) = \underline{E}^1 e^{-t} \quad \text{or} \quad \underline{e}(t) = \underline{E}^2 e^{-1/3 t}$$

Because system is __linear__ and __homogeneous__, any linear combination of these is a solution. Therefore, most general solution is

$$\underline{e}(t) = a\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t} + b\begin{bmatrix}1\\2/3\\1\end{bmatrix}e^{-\frac{1}{3}t}$$

where a, b are constants chosen to satisfy initial conditions:

$$\left.\begin{aligned} e_1(0) &= -a + b \\ e_3(0) &= a + b \end{aligned}\right\} \Rightarrow \quad \begin{aligned} a &= \frac{e_3(0) - e_1(0)}{2} \\ b &= \frac{e_3(0) + e_1(0)}{2} \end{aligned}$$

So,

$$e_1(t) = \frac{e_1(0) - e_3(0)}{2} e^{-t} + \frac{e_1(0) + e_3(0)}{2} e^{-t/3}$$

$$e_2(t) = \frac{2}{3}\,\frac{e_1(0) + e_3(0)}{2} e^{-t/3}$$

$$e_3(t) = \frac{e_3(0) - e_1(0)}{2} e^{-t} + \frac{e_3(0) + e_1(0)}{2} e^{-t/3}$$

<u>Whew</u>!!