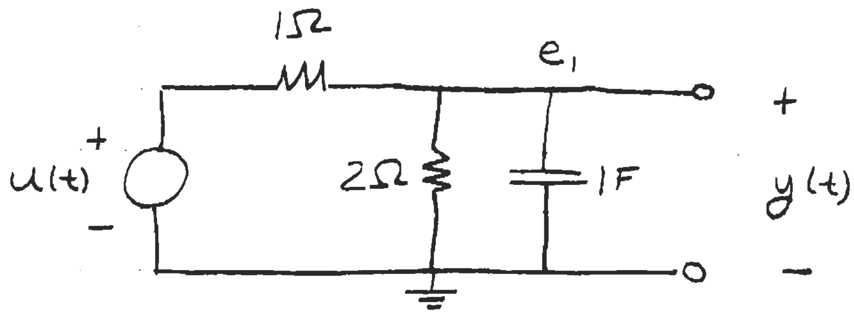
Lecture S5

Example What is response of system



to input $u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Solution First, need to find step response, $g_s(t)$.

Use node method to write differential equation

$$e_1: \quad \frac{e_1 - u}{1} + \frac{e_1 - 0}{2} + 1 \cdot \frac{d}{dt}(e_1 - 0) = 0$$

$$\Rightarrow \boxed{\dot{e}_1 + 1.5\, e_1 = u}$$

To find step response, need homogeneous and particular solution

homogeneous $e_1 = E e^{st}, \quad u = 0$

$$\Rightarrow \cancel{E} s \cancel{e^{st}} + 1.5 \cancel{E} \cancel{e^{st}} = 0$$

$$\Rightarrow s + 1.5 = 0$$

$$\Rightarrow s = -1.5$$

$$\Rightarrow e_1(t) = E e^{-1.5t} \quad \text{(homogeneous)}$$

<u>particular</u>

$$\dot{e}_1 + 1.5 e_1 = 1$$

particular solution is a constant

$$\Rightarrow 0 + 1.5 \cdot e_1 = 1$$

$$\Rightarrow e_1 = \frac{1}{1.5} = 2/3$$

$$\Rightarrow e_1(t) = 2/3 \quad \text{(particular)}$$

<u>Total solution:</u>

$$e_1(t) = 2/3 + E e^{-1.5t}$$

Initial condition:

$$e_1(0) = 0 \quad [\text{capacitor uncharged at } t=0]$$

$$e_1(0) = 2/3 + E e^{-1.5\cdot 0} = 2/3 + E = 0$$

$$E = -2/3$$

$$\Rightarrow g_s(t) = \begin{cases} 2/3 - 2/3\, e^{-1.5t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= \left(2/3 - 2/3\, e^{-1.5t}\right) \sigma(t)$$

makes $g_s(t) = 0$ for $t < 0$.

<u>Impulse Response</u>:

$$g(t) = \frac{d}{dt} g_s(t)$$

$$= 0, \quad t < 0$$

$$= (-2/3)(-1.5)\, e^{-1.5t}$$

$$= e^{-1.5t}, \quad t \geq 0$$

$$\Rightarrow g(t) = \begin{cases} e^{-1.5t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= e^{-1.5t} \sigma(t)$$

Note: In this case, there is no discontinuity in $g_s(t)$ at time $t=0$. If there were, there would also be an impulse in $g(t)$ at $t=0$.

Now that we know impulse response, can find $y(t)$:

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)\, u(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-1.5(t-\tau)}\, \sigma(t-\tau)}_{g(t-\tau)}\; \underbrace{e^{-\tau}\, \sigma(\tau)}_{u(\tau)}\, d\tau$$

$$= \int_0^t e^{-1.5t}\, e^{+1.5\tau}\, e^{-\tau}\, d\tau$$

$$= e^{-1.5t} \int_0^t e^{+0.5\tau}\, d\tau$$

$$= e^{-1.5t}\, \frac{1}{0.5}\, e^{0.5\tau} \Big|_{\tau=0}^{t}$$

$$= 2e^{-1.5t}\left[e^{0.5t} - e^{0}\right]$$

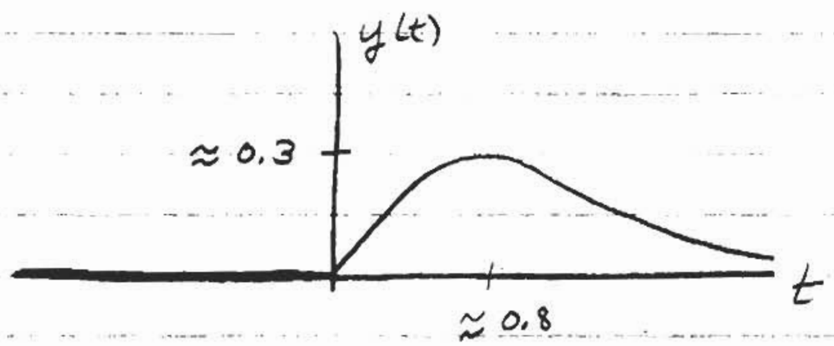
$$= 2e^{-t} - 2e^{-1.5t}, \quad t \geq 0$$

particular sol'n!

homogeneous sol'n!

$$y(t) = 0, \qquad t < 0$$

$$\Rightarrow \boxed{y(t) = \left[2e^{-t} - 2e^{-1.5t}\right]\sigma(t)}$$



Convolution

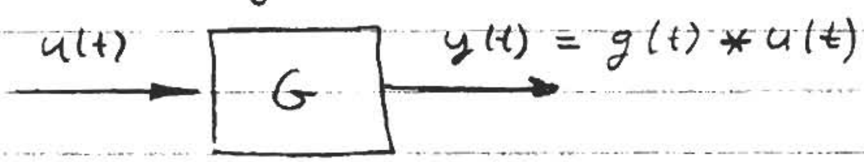
An integral of the form

$$\int_{-\infty}^{\infty} g(t-\tau)\, u(\tau)\, d\tau$$

$$\equiv g(t) * u(t)$$

convolution

So often will just write



Will talk more about convolution next time — today, we just need the notation.

For now, we need some properties of the unit impulse.