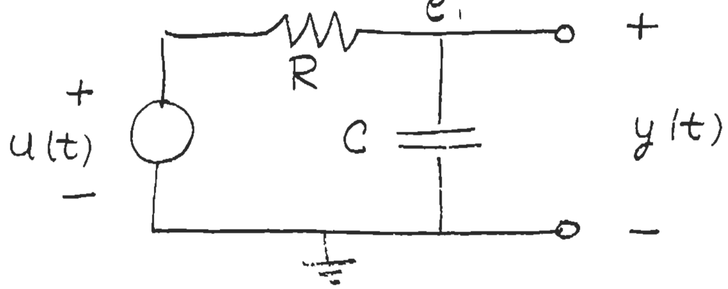
LECTURE S10

Analysis of Systems Using Laplace Transforms

Great thing about LTs is that they solve for both particular and homogeneous solutions at the same time, using only algebra!

Example Find the response of the system



$R = 2\Omega$
$C = 0.1F$

to a step input,

$$u(t) = 1V \cdot \sigma(t)$$

with initial condition

$$y(0) = 2V$$

[This is a mixed problem, with both an input and nonzero IC's]

To solve, use the node method to write the differential equation for the system:

$$C\dot{e}_1 + \frac{1}{R}(e_1 - u) = 0$$

$$\Rightarrow \quad \dot{e}_1 + \frac{1}{RC} e_1 = \frac{1}{RC} u = \frac{1}{RC} \sigma(t)$$

Now, simply LT both sides:

$$s E_1(s) - e_1(0) + \frac{1}{RC} E_1(s) = \frac{1}{RC} \frac{1}{s}$$

Solve for $E_1(s)$:

$$\left[s + 1/RC \right] E_1(s) = e_1(0) + \frac{1}{RCs}$$

$$E_1(s) = \frac{e_1(0)}{s + 1/RC} + \frac{1}{(s + 1/RC)(RCs)}$$

Plug in values:

$$E_1(s) = \frac{2}{s+5} + \frac{1}{(s+5)(0.2s)}$$

Do a "partial fraction expansion" (more next lecture):

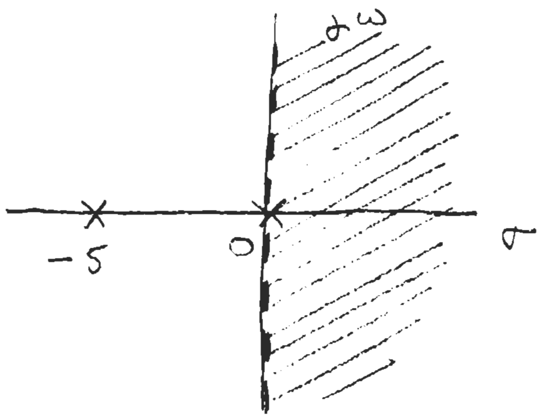
$$E_1(s) = \frac{1}{s+5} + \frac{1}{s} \qquad \text{(check this!)}$$

So,

$$\mathcal{L}[y(t)] = \mathcal{L}[e_1(t)]$$

$$= \frac{1}{s+5} + \frac{1}{s}$$

What is region of convergence?



__Why?__ Region of convergence for __unilateral__ transform must be of form

$$\mathrm{Re}[s] > \sigma_0$$

That is, if $\int_0^\infty e^{-st} g(t)\,dt$ converges for some s_1, it must converge for s_2, if $\mathrm{Re}[s_2] \geq \mathrm{Re}[s_1]$, since

$$\left|e^{-s_2 t}\right| = e^{-\mathrm{Re}[s_2]t} \leq e^{-\mathrm{Re}[s_1]t} = \left|e^{-s_1 t}\right|$$

$(t \geq 0)$

So $y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+5} + \frac{1}{s}\right]$

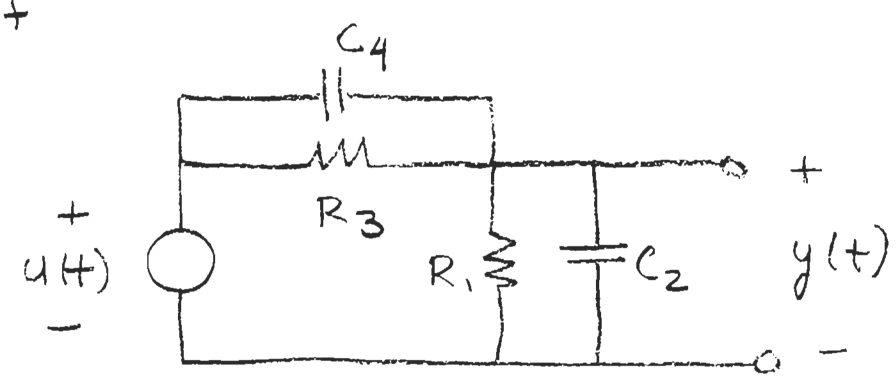
$$= \left[e^{-5t} + 1\right]\sigma(t)$$

(We can recognize these transforms by inspection)

So,

$$y(t) = \underbrace{1}_{\text{particular solution}} + \underbrace{e^{-5t}}_{\text{homogeneous solution}}, \quad t \geq 0$$

Example Find the impulse response of



$R_1 = R_3 = 2\Omega$

$C_2 = 0.4$ F

$C_4 = 0.1$ F

Use impedance methods to find $G(s)$;
$g(t) = \mathcal{L}^{-1}[G(s)]$ = impulse response

Circuit is a voltage divider:

$$G(s) = \frac{Z_1 \| Z_2}{Z_1 \| Z_2 + Z_3 \| Z_4}$$

$$Z_1 \| Z_2 = R_1 \| \frac{1}{C_2 s}$$

$$= \frac{\frac{R_1}{C_2 s}}{R_1 + \frac{1}{C_2 s}}$$

$$= \frac{R_1}{R_1 C_2 s + 1} = \frac{2}{0.8 s + 1}$$

$$Z_3 \| Z_4 = \frac{2}{0.2 s + 1}$$

$$\Rightarrow G(s) = \frac{\frac{2}{0.8 s + 1}}{\frac{2}{0.8 s + 1} + \frac{2}{0.2 s + 1}} = \frac{0.2 s + 1}{s + 2}$$

(check this!)

Do partial fraction expansion:

$$G(s) = 0.2 + \frac{0.6}{s+2}, \quad \text{(check this!)}$$

$$\mathrm{Re}[s] > -2 \quad \text{(why?)}$$

So

$$g(t) = \mathcal{L}^{-1}[G(s)] = 0.2\,\delta(t) + 0.6\,e^{-2t}\,\sigma(t)$$