Block 3 -Materials and Elasticity
Lecture M17: Engineering Elastic Constants

There are three purposes to this block of lectures:

1. To complete our quick journey through continuum mechanics, to provide you with a continuum version of a constitutive law - at least for linear elastic materials

$$\sigma_{pq} = E?\varepsilon_{mn}$$
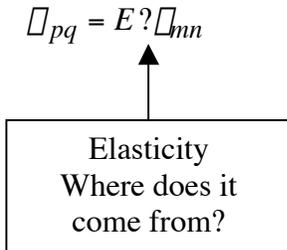
| Elasticity |
| Where does it |
| come from? |

2. Increasingly, materials are designed along with the structure, you need insight into what contributes to material properties. What you can control. What you cannot. This will also allow us to understand the limits of the model of linear elasticity for a material.

3. To allow you to select quantitatively materials for applications as part of the design process.

The lectures associated with objectives 2 and 3 will closely follow Ashby and Jones chapters 1-7. This is an excellent reference and will not be supplemented by web-posted notes. The notes for the lectures associated with objective 1 are reproduced here.

Engineering Elastic Properties of Materials

In order to understand how we link stress and strain we need to understand that there are two points of view to this matter. There is the experimental point of view that some properties (behaviors) are easier to measure than others, and there is the mathematical point of view that some representations of physical phenomena are mathematically easier to handle than others. In the present case, engineering elastic constants are derived from an experimental point of view, whereas the stress and strain tensors, are mathematically useful. Ultimately we need to resolve these two points of view.

Young's modulus and Poisson's ratio

From the truss and strain laboratories you are now familiar with at least two elastic constants.

If we apply a uniaxial tensile stress σ_L to a constant cross-section rod of material, we will obtain a biaxial state of strain, consisting of an axial tensile strain ε_L and a transverse strain ε_T. The axial strain will be tensile for a tensile applied stress, and the transverse strain will usually be compressive. We can measure the strains using resistance strain gauges.
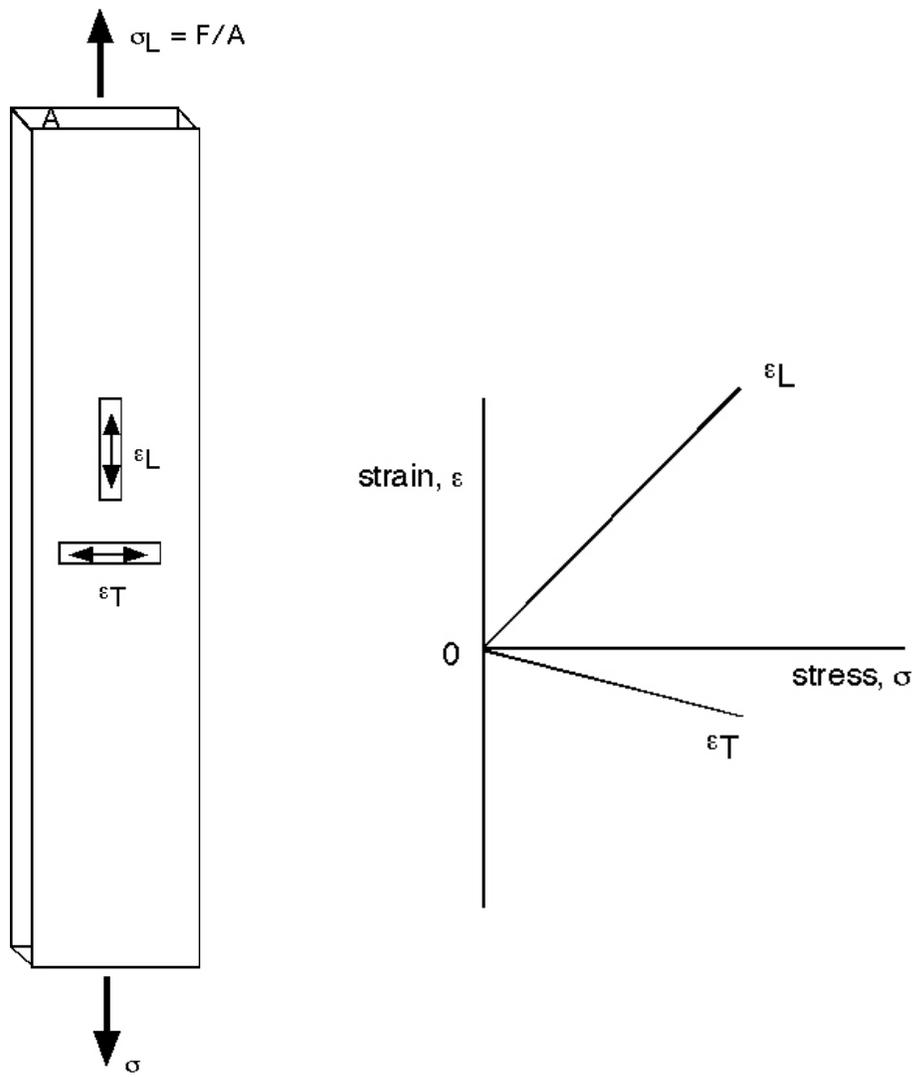
For many materials, over some range of applied stress, the applied stress and the resulting strains will follow a linear relationship. This observation is the basis for the definition of the engineering elastic constants. The Young's modulus, E, is defined as the constant of proportionality between a uniaxial applied stress and the resulting axial strain, i.e:

$$\sigma_L = E\varepsilon_L$$

Note. This only applies for a uniaxial applied stress, and the component of strain in the direction of the applied stress.

We can also define the Poisson's ratio, ν, as the ratio of the transverse strain to the axial strain. Since for the vast majority of materials the transverse strain is compressive for a tensile applied stress, the Poisson's ratio is defined as the negative of this ratio, to give a positive quantity. I.e:

$$\nu = -\frac{\varepsilon_T}{\varepsilon_L}$$

A similar process, of performing experiments in which a well-defined stress state is applied and the resulting strain state is characterized leads us to define two other elastic constants.

The Shear Modulus

Application of a state of pure shear, leads to a shear strain:
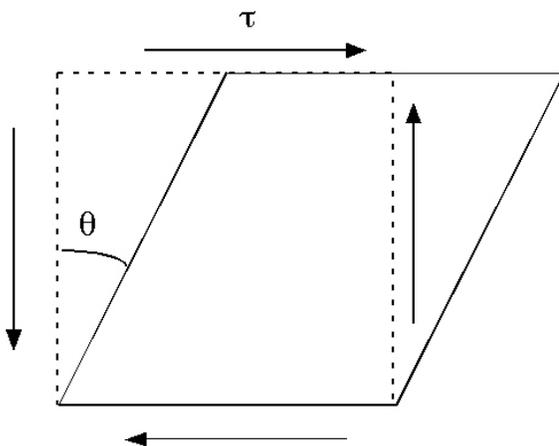
Note angles are exaggerated in the figure.

An applied shear stress leads to an applied shear strain. The shear strain, γ, is defined in engineering notation, and therefore equals the total change in angle: $\gamma = \theta$.

Consistent with the definition of the Young's modulus, the Shear modulus. G, is defined as:

$$\tau = G\gamma$$

Again, note, that this relationship only holds if a pure shear is applied to a specimen.

The Bulk Modulus

The final elastic constant that is of interest to us is that of the bulk modulus. Materials are slightly compressible. If a hydrostatic pressure, p, is applied to a volume of material, V, this will result in a slight reduction in volume, ΔV.
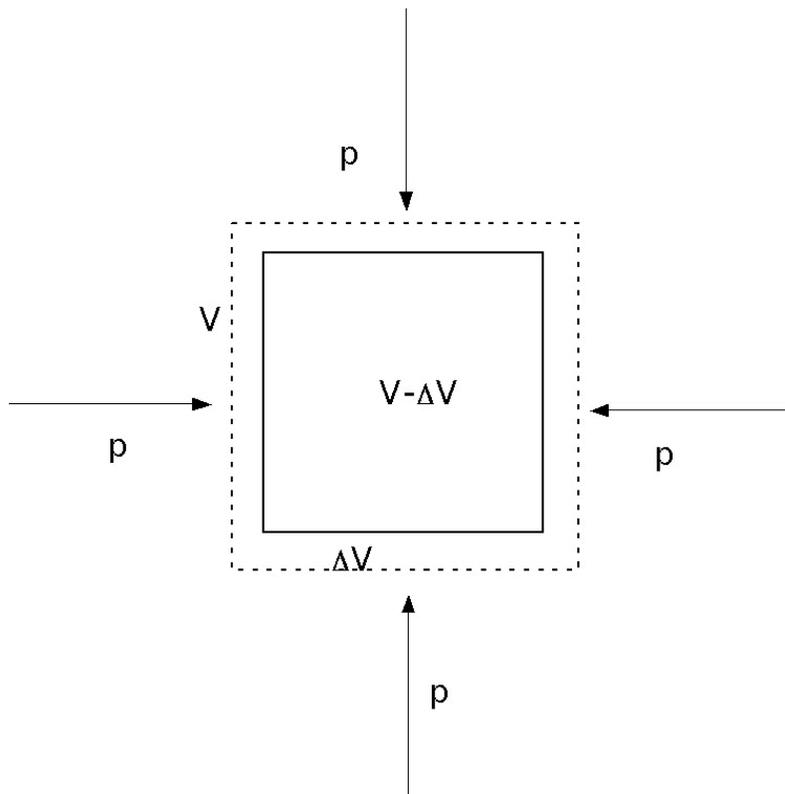
This leads to a definition of the volumetric strain, Δ: $\Delta = \dfrac{\Delta V}{V}$

Thus we can define a bulk modulus, K, as:

$$p = K\Delta$$

Note, that if the pressure is represented as a stress, it would be negative, as would the change in volume.

For reasons that will become apparent later, The Young's modulus, Shear Modulus, Bulk modulus and Poisson's ratio are linked. For most materials Poission's ratio's are approximately 0.33, and for these materials $K \approx E$ and $G \approx \dfrac{3}{8}E$. However, values of the Young's modulus can vary widely.

The Young's Moduli of Engineering Materials

A useful way of representing the range of Young's moduli is to plot them for all the classes of material against some other material property. Since we are often interested in light weight structures, we will choose the density of the materials for this purpose.
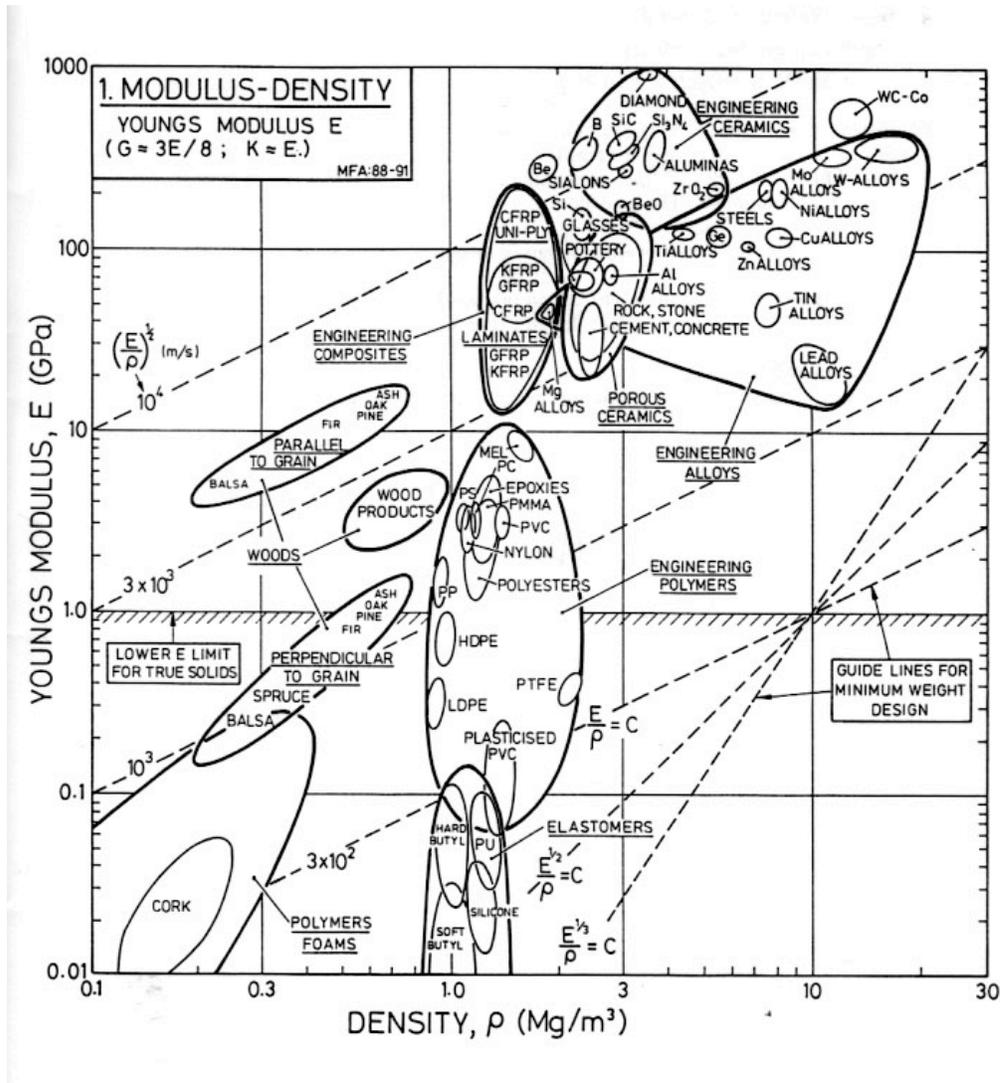
From: Material Selection in Mechanical Design, M.F Ashby, Pergamon Press, Oxford, 1992

Note, how the different classes of material tend to cluster: Metals have relatively high moduli and high densities. Polymers have low moduli and densities. Glasses and

ceramics have high moduli and somewhat lower densities than metals. There are also some materials that have quite wide ranges of moduli (and densities), while others (metals and ceramics) are relatively narrowly banded. Finally note how wide an overall range of moduli is represented, from 0.01 GPA for foams to 1000 GPA for diamond. The range of densities is somewhat less, but still spans more than two orders of magnitude.

M18 Elastic moduli of composites, anisotropic materials

We will return to better understand what leads to the moduli characteristic of different classes of material in a few lectures time.

Now let's get back to examining the elastic constants. Let us look more closely at one particular class of material, fiber composites. Reference to the material property chart above we can see that composites (CFRP – carbon fiber reinforced polymers) have higher modulus to density ratio's than many metals. Why is this?

The key is that very fine (6 μm diameter) carbon fibers can be produced with a modulus comparable to that of ceramics (200-1000 GPa). These fibers also have very tensile high strengths, much higher than normally exhibited by bulk ceramics, which tend to be brittle, and have a low strength as a result. However, they are fibers, so they cannot carry multiaxial loads on their own. However, if they are surrounded by a "matrix" to provide lateral support, and to transfer load between fibers if one fiber happens to break, they can result in materials with high moduli and strengths. Polymers such as epoxy resins are often used as matrices.

Let us examine how we can estimate the Young's modulus of the resulting composite material. Initially we will consider a two dimensional case. 2-D fibers interspersed with a 2-D matrix. The fibers have a Young's modulus E_f and the matrix a Young's modulus E_m
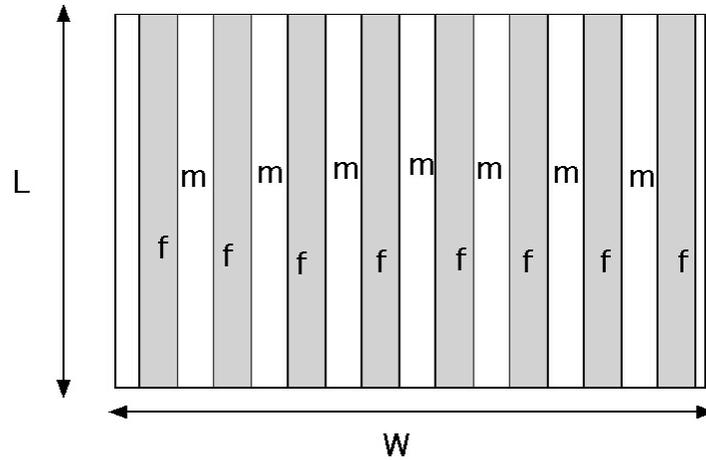
We can define a volume fraction of fibers, V_f, such that the total volume of fibers is equal to the volume fraction of fibers multiplied by the total volume of material. We can similarly define a volume fraction of matrix, V_m, which in the absence of any other materials in the composite, or porosity, is given by $V_m=1-V_f$.

Now consider what happens if the composite is loaded by force F, that results in a displacement, u, parallel to the fiber direction:
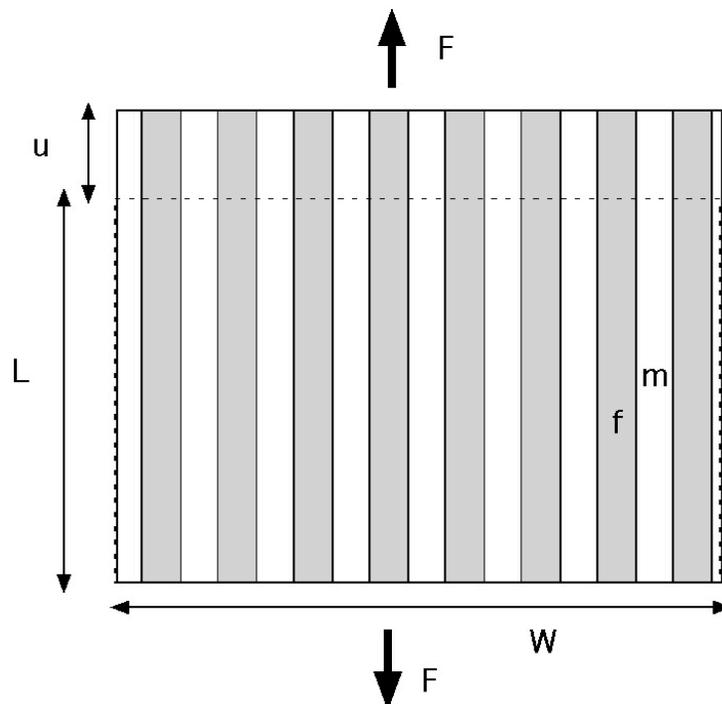
If the fiber and matrix are well bonded together, they experience the same strain in the fiber direction: $\varepsilon_L = \dfrac{u}{L}$.

However the resulting stress in the fibers and matrix is different because they have different Young's moduli:

$$\sigma_f = E_f \varepsilon_L \quad \text{and} \quad \sigma_m = E_m \varepsilon_L$$

The total force applied to the composite specimen must be in equilibrium with the total force due to the stresses in the fiber and matrix. The internal force is given by multiplying the stress by the area of fiber or matrix it acts on. If the total cross-sectional area of the composite specimen is A, then the cross-sectional area of fibers and matrix are given by the total area multiplied by the volume fraction of fiber or matrix. Thus the force carried by the fibers and matrix are given by:

$$F = V_f A \sigma_f + V_m A \sigma_m$$

Substituting in for the stress in terms of the strain we obtain:

$$F = V_f A E_f \varepsilon_L + \left(1 - V_f\right) A E_m \varepsilon_L$$

Rearranging and dividing by the cross sectional area gives the average longitudinal stress carried by the composite in terms of the longitudinal strain:

$$\sigma_L = \left(V_f E_f + \left(1 - V_f\right) E_m\right) \varepsilon_L$$

From which we can see that the modulus of the composite parallel to the fiber direction is given by

$$E_L = \left(V_f E_f + \left(1 - V_f\right) E_m\right)$$

Now consider what happens if the composite specimen is loaded perpendicular to the fiber direction:
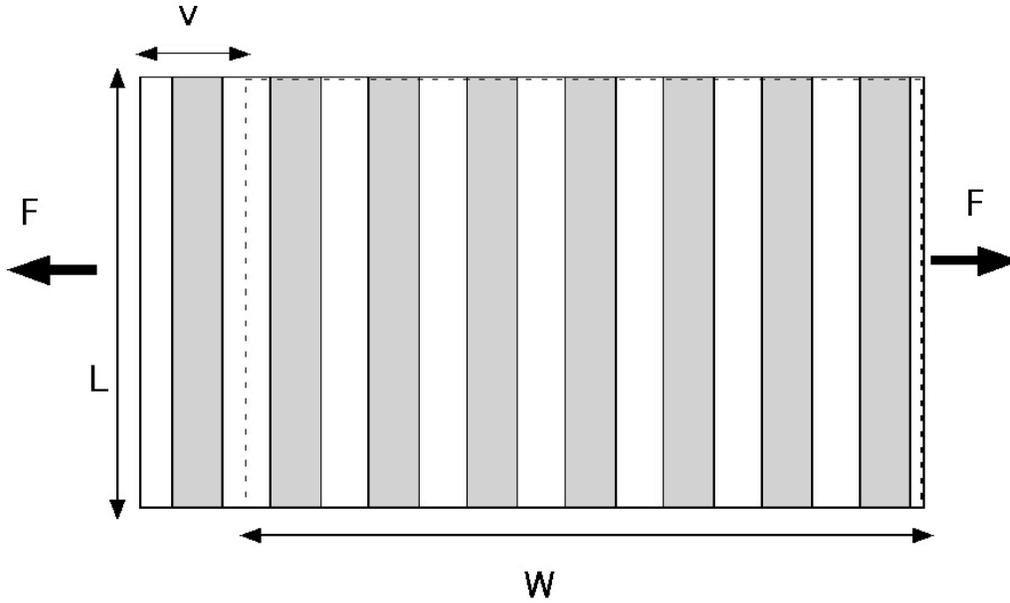
Now the load must be carried equally by the fibers and matrix, but the fibers and matrix will experience different strains, $\varepsilon_m \ and \ \ \varepsilon_f$. The strains lead to deformations that must sum to give the total elongation, v, of the composite:

Hence;
$$v = \varepsilon_m\left(1 - V_f\right)W + \varepsilon_f V_f W$$

Given that the strain in the matrix and fibers depend on the stress in the matrix and fibers, and their Young's moduli. We obtain:

$$v = \frac{\sigma}{E_m}\left(1 - V_f\right)W + \frac{\sigma}{E_f}V_f W$$

Dividing through by the gauge length, W gives the total strain in the composite,

$$\varepsilon_T = \frac{\sigma}{E_m}\left(1 - V_f\right) + \frac{\sigma}{E_f}V_f$$

From which we can see that the Young's modulus of the composite transverse to the fiber direction is given by:

$$E_T = \frac{1}{\dfrac{\left(1 - V_f\right)}{E_m} + \dfrac{V_f}{E_f}}$$

Which is different from the Young's modulus parallel to the fiber direction. Thus fiber composites are an example of a material that has different properties in different directions. This is termed **"anisotropy"** and most fiber composites are **"anisotropic".** Materials which have the same properties in all directions are termed **"isotropic".**

Note, that the estimate for the Young's modulus of a fiber composite parallel to the fiber direction is very good, however, the estimate for the Young's modulus perpendicular to the fiber direction underestimates the value you would measure experimentally.

Since we are interested in composite materials for many structural applications, we would like to have a method for linking general stress and strain that can account for anisotropy. So back to continuum elasticity.

M19 Generalized Hooke's Law

We have met the engineering elastic constants, Young's moduli, Shear Moduli and Poisson's ratio's, and understand that many structural materials behave elastically over some range of stress and strain.

Now we want to add a mathematical formalism to this physical basis, i.e. our 3rd great principle, that of constitutive behavior.

A couple of problems we would like to be able to solve:
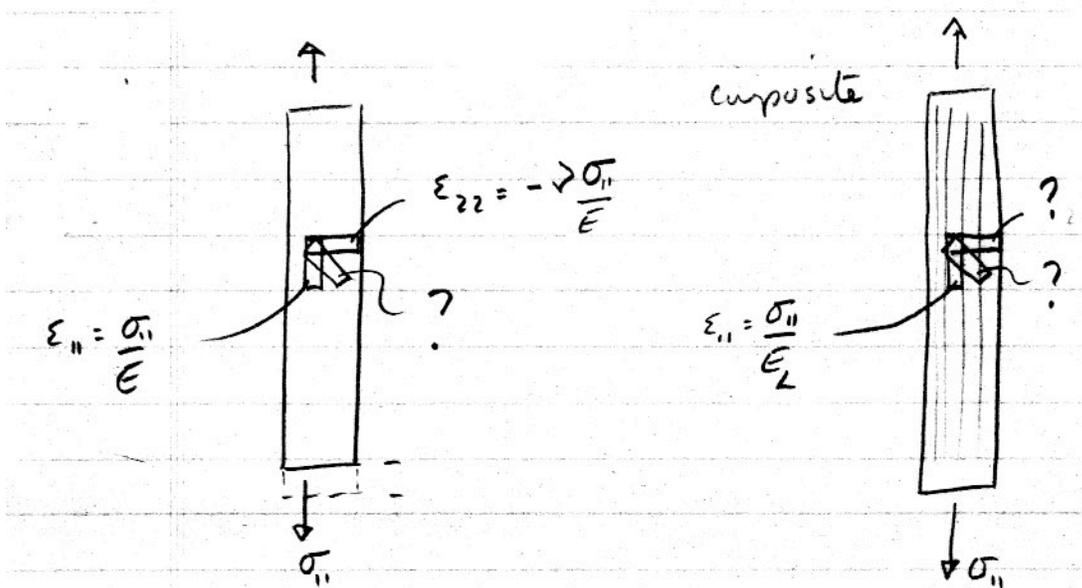
We would also like to be able to deal with any state of multiaxial stress and convert to the resulting strains, or vice versa. To do this we need to revisit tensor stress and strain.

i.e. we want the elastic property that links the stress tensor to the strain tensor:

$$\sigma_{mn} = E_{mnpq}\varepsilon_{pq}$$

Where E_{mnpq} is the 4th order (i.e. 4 subscripts) ELASTICITY (or STIFFNESS) tensor.

e.g:

$$\sigma_{11} = E_{1111}\varepsilon_{11} + E_{1112}\varepsilon_{12} + E_{1113}\varepsilon_{13} \qquad (p = 1, sum\ on\ q)$$

$$+E_{1121}\varepsilon_{21} + E_{1122}\varepsilon_{22} + E_{1123}\varepsilon_{23} \qquad (p = 2, sum\ on\ q)$$

$$+E_{1131}\varepsilon_{31} + E_{1132}\varepsilon_{32} + E_{1133}\varepsilon_{33} \qquad (p = 3, sum\ on\ q)$$

4th order tensor has 81 components, m,n,p,q =1, 2 and 3 therefore $3^4 = 81$ terms

But fortunately there are symmetries, so there are fewer independent terms

1. $\sigma_{mn} = \sigma_{nm}$ (symmetry of the stress tensor - due to equilibrium of moments)

this implies that: $E_{mnpq} = E_{nmpq}$

2. $\varepsilon_{mn} = \varepsilon_{nm}$ (symmetry of the strain tensor - due to definition of strain tensor - geometrical considerations)

this implies that: $E_{mnpq} = E_{mnqp}$

3. From thermodynamic considerations (first law) $E_{mnpq} = E_{pqmn}$

Also note that since $\sigma_{mn} = \sigma_{nm}$ so the nine separate equations represented by: $\sigma_{mn} = E_{mnpq}\varepsilon_{pq}$ reduce to six.

And since $\varepsilon_{mn} = \varepsilon_{nm}$ and $E_{mnpq} = E_{mnqp}$, terms such as $E_{mnpq}\varepsilon_{pq} + E_{mnqp}\varepsilon_{qp} = 2E_{mnpq}\varepsilon_{pq}$

With all of these considerations we end up with only (!) 21 independent components of the elasticity tensor. In matrix form, this can be written as:

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1113} & 2E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & 2E_{2223} & 2E_{2213} & 2E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & 2E_{3323} & 2E_{3313} & 2E_{3312} \\ E_{1123} & E_{2223} & E_{3323} & 2E_{2323} & 2E_{1323} & 2E_{1223} \\ E_{1113} & E_{2213} & E_{3313} & 2E_{1323} & 2E_{1313} & 2E_{1213} \\ E_{1112} & E_{2212} & E_{3312} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}
$$

Components of E_{mnpq} can be placed into 3 groups according to their physical significance:

$$
\begin{Bmatrix} E_{1111} & E_{1122} \\ E_{2222} & E_{1133} \\ E_{3333} & E_{2233} \end{Bmatrix}
$$ Link extensional strains to extensional stresses

$$
\begin{Bmatrix} E_{1212} & E_{1213} \\ E_{1313} & E_{1323} \\ E_{2323} & E_{1223} \end{Bmatrix}
$$ Link shear strains to shear stresses

$$
\begin{Bmatrix} E_{1112} & E_{2212} & E_{3312} \\ E_{1113} & E_{2213} & E_{3313} \\ E_{1123} & E_{2223} & E_{3323} \end{Bmatrix}
$$ Coupling terms, link extensional strains to shear stresses

and shear strains to extensional stresses.

Since it links strains to stresses, E_{mnpq} is also termed the "stiffness tensor". By inverting the matrix we could also obtain the tensor linking stresses to strains, i.e. $\varepsilon_{mn} = S_{mnpq} \sigma_{pq}$

Where S_{mnpq} is the "Compliance" tensor (Why S is for compliance and E is used for stiffness is unclear to me!)

Even with the simplifications, 21 independent terms seems rather too many to have to deal with!. Let's go back to the engineering elastic constants and see if we can see how to simplify this list further.

We know that there are several different classes of material. Most metals and ceramics are **isotropic**, that is they have the same properties in any direction that you measure. By contrast, fiber-reinforced composites may have different properties in different directions, i.e. they are **anisotropic**.

Elasticity of Isotropic Materials

Let's start with the simplest case of an isotropic material that is loaded by all possible components of stress and we want to know the resulting strains. Also let's ignore thermal expansion strains for the time being. We have six components of stress producing six components of strain, therefore we need a six by six matrix

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ & & & & & \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ & & & & & - \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix}
$$

We also know that for small strains, and elastic materials the contributions of the separate components of stress will superimpose. So let's consider the case of only σ_x applied and all the other components of strain are zero:

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ & & & & & \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ & & & & & - \end{pmatrix} \begin{pmatrix} \sigma_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Taking each component in turn:

What does ε_x equal?. This just reduces to a 1-D tensile test, so $\varepsilon_x = \dfrac{\sigma_x}{E}$

And ε_y , ε_z are given by the Poisson contractions, so: $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\dfrac{\sigma_x}{E}$

This allows us to fill in the first line of the matrix, and also by noticing that we could have equally well applied σ_y or σ_z, and obtained similar relationships, we can fill in all of the top left hand quadrant of the matrix:

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}
=
\left(
\begin{array}{ccc|ccc}
\dfrac{1}{E} & \dfrac{-\nu}{E} & \dfrac{-\nu}{E} & - & - & - \\
\dfrac{-\nu}{E} & \dfrac{1}{E} & \dfrac{-\nu}{E} & - & - & - \\
\dfrac{-\nu}{E} & \dfrac{-\nu}{E} & \dfrac{1}{E} & - & - & - \\
- & - & - & - & - & - \\
- & - & - & - & - & - \\
- & - & - & - & - & -
\end{array}
\right)
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix}
$$

If instead of applying an extensional stress we applied a shear stress, we know that the shear stress and shear strain are linked by the shear modulus, so:

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}
=
\left(
\begin{array}{ccc|ccc}
\dfrac{1}{E} & \dfrac{-\nu}{E} & \dfrac{-\nu}{E} & - & - & - \\
\dfrac{-\nu}{E} & \dfrac{1}{E} & \dfrac{-\nu}{E} & - & - & - \\
\dfrac{-\nu}{E} & \dfrac{-\nu}{E} & \dfrac{1}{E} & - & - & - \\
- & - & - & \dfrac{1}{G} & 0 & 0 \\
- & - & - & 0 & \dfrac{1}{G} & 0 \\
- & - & - & 0 & 0 & \dfrac{1}{G}
\end{array}
\right)
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix}
$$

Finally we note that for isotropic materials the application of an extensional stress does not result in a shear strain or vice versa, so the top right and bottom left quadrants are populated by zeros:

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{E} & \dfrac{-v}{E} & \dfrac{-v}{E} & 0 & 0 & 0 \\[2mm] \dfrac{-v}{E} & \dfrac{1}{E} & \dfrac{-v}{E} & 0 & 0 & 0 \\[2mm] \dfrac{-v}{E} & \dfrac{-v}{E} & \dfrac{1}{E} & 0 & 0 & 0 \\[2mm] 0 & 0 & 0 & \dfrac{1}{G} & 0 & 0 \\[2mm] 0 & 0 & 0 & 0 & \dfrac{1}{G} & 0 \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix}
$$

We have three separate elastic constants required, i.e: E, v and G. However if we go back to our knowledge of stress and strain transformation we can reduce this further. Remember that the application of a shear stress can be thought of as a shear stress resulting in a shear strain:

$$ \gamma = \frac{\tau}{G} $$

or a biaxial stress state, of a combined tension and compression, at 45 degrees to the axis of pure shear, i.e:
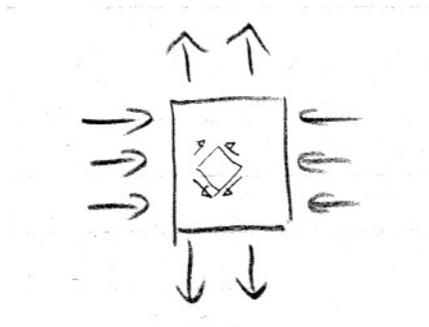
In terms of the Mohr's circles of stress and strain these appear as (note the factor of two for the representation of shear strain on the Mohr's circle):
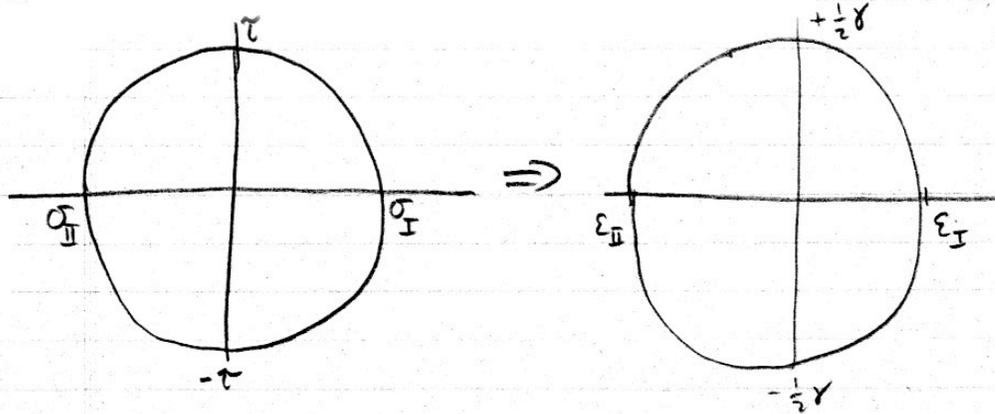
For the biaxial tensile and compressive stress the resulting (principal) strain is given by:
(see CDL 5.4)

$$\varepsilon_I = \frac{\sigma_I}{E} - v\frac{\sigma_{II}}{E} \qquad \boxed{\sigma_{II} = -\sigma_I}$$

but for the case of pure shear: $\varepsilon_I = \frac{1}{2}\gamma$ (From Mohr's Circle -remember the

factor of two between tensor and engineering shear strain)

and:

$$\tau = \sigma_{II} = -\sigma_I$$

$$\therefore \gamma = 2\left(\frac{1}{E} + \frac{v}{E}\right)\tau \Rightarrow G = \frac{E}{2(1+v)}$$

So we actually only have two independent elastic constants for an isotropic material.
Note that this only applies for isotropic materials

If we want to go in the reverse direction (i.e. have known strains and want to calculate stresses) we need to invert the matrix of elastic constants. Note, this situation may arise because we can experimentally measure strains using strain gauges. The inverse matrix is usually expressed in terms of groupings of the elastic constant, known as Lamé's constants, μ & λ, where:

$$\mu = \frac{E}{2(1+v)} = G$$

$$\lambda = \frac{vE}{(1+v)(1-2v)}$$

Thus:
$$
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}
=
\begin{pmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{pmatrix}
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix}
$$

We can also include the effect of thermal expansion:

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}
=
\begin{pmatrix}
\frac{1}{E} & -\frac{v}{E} & -\frac{v}{E} & 0 & 0 & 0 \\
-\frac{v}{E} & \frac{1}{E} & -\frac{v}{E} & 0 & 0 & 0 \\
-\frac{v}{E} & -\frac{v}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{pmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix}
+
\begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}
\Delta T
$$

Can now relate back to the elasticity tensor that we started with, remembering that engineering shear strain is defined as being twice tensor shear strain, i.e. $\gamma_{xy} = 2\varepsilon_{12}$ etc.:

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
=
\begin{bmatrix}
E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1113} & 2E_{1112} \\
E_{1122} & E_{2222} & E_{2233} & 2E_{2223} & 2E_{2213} & 2E_{2212} \\
E_{1133} & E_{2233} & E_{3333} & 2E_{3323} & 2E_{3313} & 2E_{3312} \\
E_{1123} & E_{2223} & E_{3323} & 2E_{2323} & 2E_{1323} & 2E_{1223} \\
E_{1113} & E_{2213} & E_{3313} & 2E_{1323} & 2E_{1313} & 2E_{1213} \\
E_{1112} & E_{2212} & E_{3312} & 2E_{1223} & 2E_{1213} & 2E_{1212}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}
$$

$$E_{1111} = E_{2222} = E_{3333} = \lambda + 2\mu$$

$$E_{1122} = E_{1133} = E_{2233} = \lambda$$

and $E_{1212} = E_{1313} = E_{2323} = \mu = G$

All other terms are equal to zero.

Note that E_{1111} does not simply equal E, the Young's modulus, but

$$E_{1212} = E_{1313} = E_{2323} = G$$

Elasticity for Non-Isotropic Materials

For more elastically complicated materials more of the elastic constants take on different values according to the direction in which they are measured. This is known as *anisotropy*. We will consider two important cases of composites:

Transversely Isotropic Materials

In M18 we estimated the elastic moduli of a composite reinforced by unidirectional fibers by assuming that they carried equal strain in the fiber direction, but equal stress in the transverse direction.

If we think about the physical situation we will see that we can define a longitudinal direction along the axis of the fibers and a plane that is transverse to the fibers. The properties in the plane transverse to the fibers are isotropic, but can differ significantly from those in the axial direction.

E.g. in the diagram shown below, the modulus in the x_1 direction, i.e. the longitudinal direction, is given by the rule of mixtures estimate, i.e: $E_L = V_f E_f + V_m E_m$
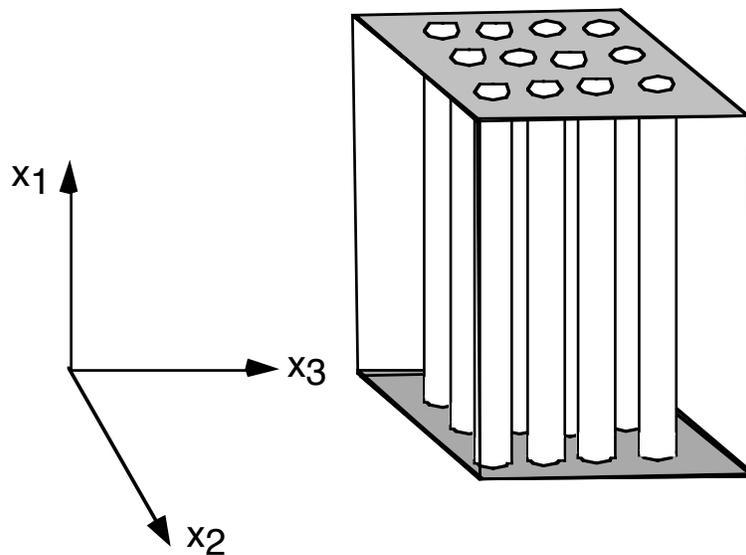
And the modulus in the x_2 and x_3 direction, or for that matter any direction in the x_2-x_3 plane (the *transverse* plane), is the same and might be estimated by the inverse rule of mixtures, i.e.: $E_T = \dfrac{1}{\left\{\dfrac{V_f}{E_f} + \dfrac{V_m}{E_m}\right\}}$

There will clearly be a Poisson's contraction in the transverse direction, due to an applied strain in the longitudinal direction, and vice versa, so we can define:

$\varepsilon_2 = \varepsilon_3 = -v_{LT}\varepsilon_1$ for an applied strain, $\varepsilon_1 = \dfrac{\sigma_1}{E_L}$

and $\varepsilon_1 = -v_{TL}\varepsilon_2 = -v_{TL}\varepsilon_3$ for applied strains of $\varepsilon_2 = \dfrac{\sigma_2}{E_T}$ or $\varepsilon_3 = \dfrac{\sigma_3}{E_T}$ respectively.

Also there would be a Poisson contraction in the x_2 direction due to an applied strain in the x_3 direction and vice versa, i.e.:

$\varepsilon_2 = -v_{TT}\varepsilon_3$ and $\varepsilon_3 = -v_{TT}\varepsilon_2$ for applied strains respectively ε_3 and ε_2 respectively.

There must also be a shear modulus between the longitudinal direction and the transverse direction, and another one in the transverse plane. i.e.

$\tau_{12} = 2G_{LT}\varepsilon_{12}$ note the factor of two due to the definition of engineering vs. tensor strain

$\tau_{13} = 2G_{LT}\varepsilon_{13}$

$\tau_{23} = 2G_{TT}\varepsilon_{23}$

But we know that the material is isotropic in the transverse plane, therefore:

$G_{TT} = \dfrac{E_T}{2(1 + v_{TT})}$

Furthermore, due to thermodynamic considerations (it would be a violation of the first law if this was not the case).:

$v_{LT}E_T = v_{TL}E_L$

This leaves us with five independent elastic constants: E_L, E_T, G_{LT}, v_{LT}, and v_{TT} in matrix form for the material orienation shown we would have:

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}
=
\begin{bmatrix}
\dfrac{1}{E_L} & \dfrac{-\nu_{LT}}{E_L} & \dfrac{-\nu_{LT}}{E_L} & 0 & 0 & 0 \\
\dfrac{-\nu_{TL}}{E_T} & \dfrac{1}{E_T} & \dfrac{-\nu_{TT}}{E_T} & 0 & 0 & 0 \\
\dfrac{-\nu_{TL}}{E_T} & \dfrac{-\nu_{TT}}{E_T} & \dfrac{1}{E_T} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{1}{2G_{TT}} & 0 & 0 \\
0 & 0 & 0 & 0 & \dfrac{1}{2G_{LT}} & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{1}{2G_{LT}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}
$$

As for an isotropic material we could also add on a vector with the strains due to thermal expansion, and we could invert the matrix to obtain the stiffness form.

Orthotropic Materials.

The second kind of non-isotropic materials that is of particular interest to aerospace structures are "Orthotropic Materials". Composite materials are rarely used as unidirectional material, the transverse properties (particularly strength and stiffness) are insufficient for most structural applications. Instead unidirectional layers (plies) are often combined to form laminates (rather like ply wood), e.g: the three ply composite shown below, orientated in an x, y, z rectangular cartesian coordinate system:
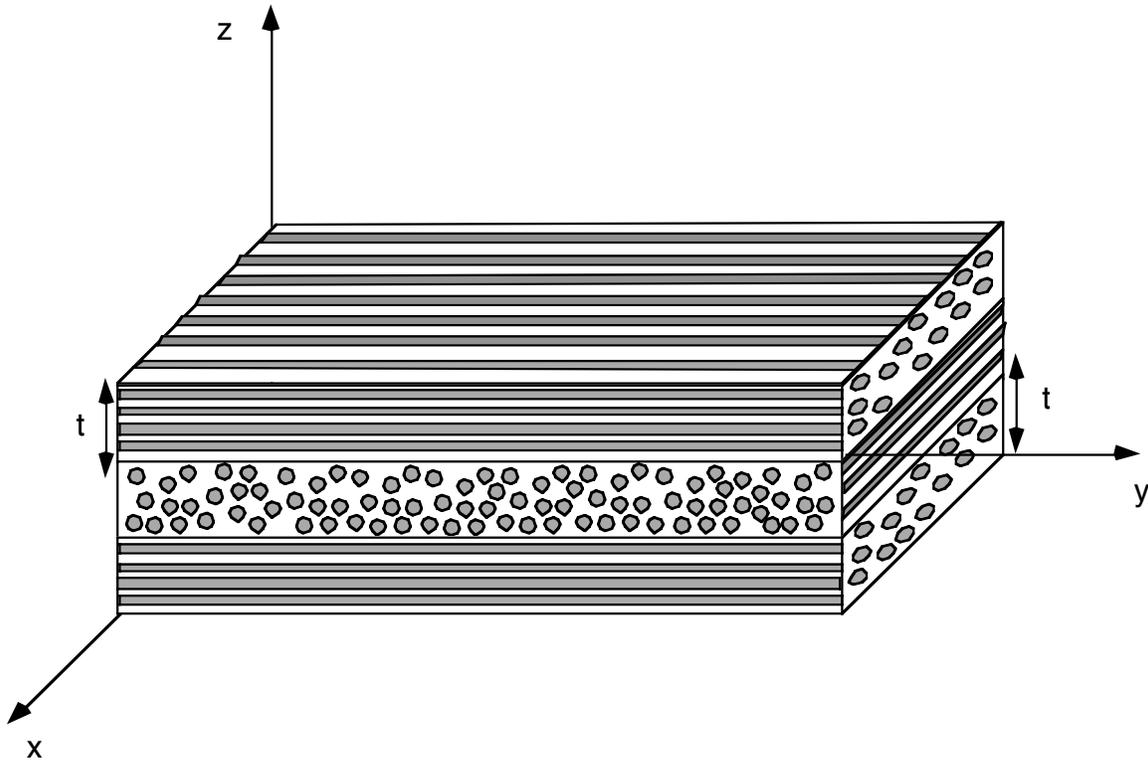
We can estimate the Young's moduli, much as we did for the unidirectional material. If the longitudinal and transverse moduli are E_L and E_T respectively, then we might expect:

$$E_x \approx \frac{1}{3}E_L + \frac{2}{3}E_T, \quad E_y \approx \frac{2}{3}E_L + \frac{1}{3}E_T, \quad E_z \approx E_T$$

i.e. the properties are different in all three orthogonal directions. Such a material is termed orthotropic. By similar reasoning to that applied to the transversely isotropic case it requires 9 independent elastic constants to define an orthotropic material: 3 Young's moduli, three Shear Moduli and three independent Poisson's ratio's ie.:

Longitudinal Moduli: $E_x = \dfrac{\sigma_x}{\varepsilon_x}, \quad E_y = \dfrac{\sigma_y}{\varepsilon_y}, \quad E_z = \dfrac{\sigma_z}{\varepsilon_z}$ for σ_m applied only

Poisson's ratios:

$$v_{xy} = \frac{-\varepsilon_y}{\varepsilon_x}, \quad v_{yx} = \frac{-\varepsilon_x}{\varepsilon_y}, \quad v_{zy} = \frac{-\varepsilon_y}{\varepsilon_z}, \quad v_{yz} = \frac{-\varepsilon_z}{\varepsilon_y}, \quad v_{zx} = \frac{-\varepsilon_x}{\varepsilon_z}, \quad v_{xz} = \frac{-\varepsilon_z}{\varepsilon_x}$$

v_{nm} represents Poisson's ratio for transverse strain in x_m direction when stressed in x_n direction

Reciprocity: $v_{nm}E_m = v_{mn}E_n$ (for n, m = x, y, z)

Shear Moduli:
$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}}, \quad G_{xz} = \frac{\tau_{xz}}{\gamma_{xz}}, \quad G_{yz} = \frac{\tau_{yz}}{\gamma_{yz}}$$

And these can be represented in matrix form as:

Orthotropic (Compliance form):

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}
=
\begin{bmatrix}
\dfrac{1}{E_x} & \dfrac{-\nu_{xy}}{E_x} & \dfrac{-\nu_{xz}}{E_x} & 0 & 0 & 0 \\
\dfrac{-\nu_{yx}}{E_y} & \dfrac{1}{E_y} & \dfrac{-\nu_{yz}}{E_y} & 0 & 0 & 0 \\
\dfrac{-\nu_{zx}}{E_z} & \dfrac{-\nu_{zy}}{E_z} & \dfrac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{1}{G_{yz}} & 0 & 0 \\
0 & 0 & 0 & 0 & \dfrac{1}{G_{xz}} & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{1}{G_{xy}}
\end{bmatrix}
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}
$$

There are other classes of elastic anisotropy that can be distinguished according to how many elastic constants are required to describe them. Fortunately we generally do not encounter fully anisotropic materials in practice. In all cases the engineering elastic constants (Young's moduli, Shear Moduli, Poisson's ratios) can be equated to the components of the stiffness and compliance tensors.

Degrees of Anisotropy:

Generally Anisotropic   21 Independent elastic constants

Monoclinic   13 Independent elastic constants

Generally Orthotropic 9 Independent elastic constants (3 E's, 3 ν's, 3 G's)

Transversely Isotropic   5 Independent elastic constants (2 E's, 2 ν's, 1 G)

Cubic   3 Independent elastic constants
Isotropic   2 Independent elastic constants (E, ν)