Department of Aeronautics & Astronautics, M.I.T.
16.001 - Materials & Structures

Quiz No. 5

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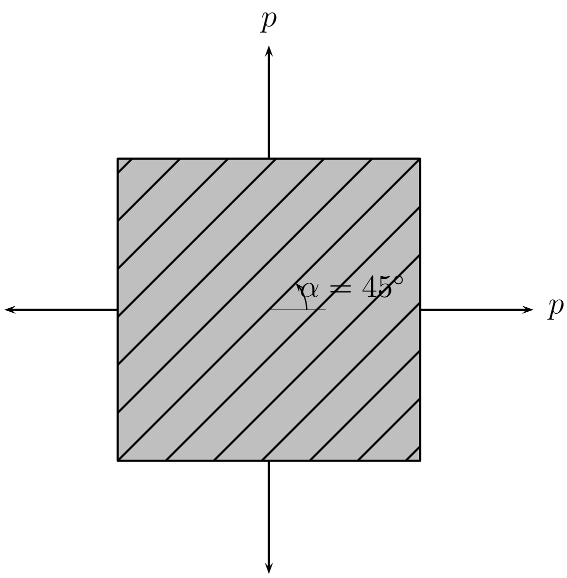
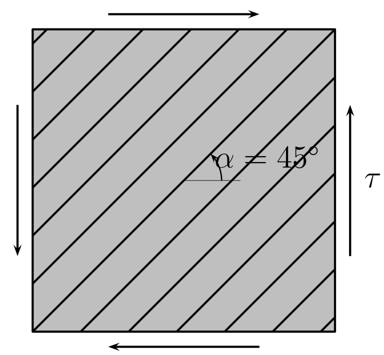
Student's name:________________________________

Question	Points	Score
1	20	
2	20	
Total:	40	

Letter grade: __________

Question 1 [20 points]
An orthotropic composite material with elastic constants $E_1, E_2, \nu_{12}, G_{12}$ is subject to two different states of stress: 1) hydrostatic, 2) and pure shear, as shown in the figures below. The dominant fiber direction (E_1) is at an angle $\alpha = 45°$ with respect to the $\mathbf{e}_1$ axis.



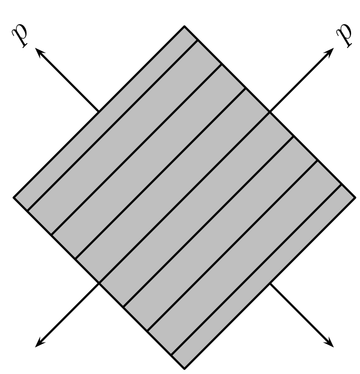


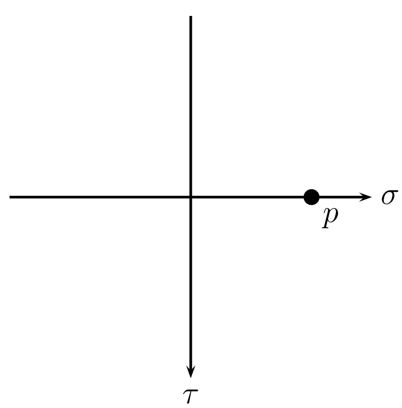
1.1 (10 points) Hydrostatic state of stress: Compute the state of strain in some axes of your choosing. Then find the principal strains and directions. In addition, prove that the maximum shear strains are independent of the Poisson ratios ν_{12}, ν_{21}

Solution: The state of stress in the given axes is:

$$[\sigma] = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$$

Since it's a hydrostatic state, all directions are principal, there is no shear stress on any axes, Mohr's circle is a point on the σ axis at coordinate p, and the stress components are the same for all axes orientations, including of course the material axes directions. Graphically,





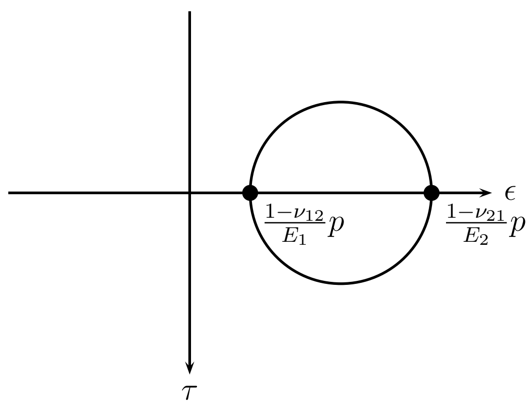
The transformed stress components are then

$$\tilde{\sigma}_{11} = p = \sigma_I$$
$$\tilde{\sigma}_{22} = p = \sigma_{II}$$
$$\tilde{\sigma}_{12} = 0$$

We can now use the constitutive law, as the stress components are in the direction aligned with the fibers:

$$\epsilon_{11} = \frac{1}{E_1}\left(p - \nu_{12}p\right) = \frac{1-\nu_{12}}{E_1}p$$
$$\epsilon_{22} = \frac{1}{E_2}\left(p - \nu_{21}p\right) = \frac{1-\nu_{21}}{E_2}p$$
$$2\epsilon_{12} = \frac{0}{G_{12}}$$
$$[\epsilon] = \begin{pmatrix} \frac{1-\nu_{12}}{E_1}p & 0 \\ 0 & \frac{1-\nu_{21}}{E_2}p \end{pmatrix}$$

Those are then principal directions of strain as well, then. Mohr's circle for strain in this case is:



The maximum shear strains are twice the radius of the circle:

$$R = |\epsilon_I - \epsilon_{II}| = \frac{1-\nu_{12}}{E_1}p - \frac{1-\nu_{21}}{E_2}p = \frac{1}{E_1} - \frac{1}{E_2} \ |p|$$

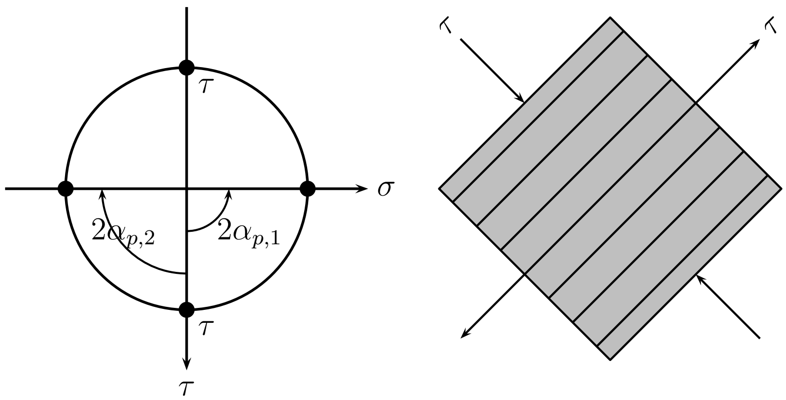
which is independent of the Poisson ratios.

1.2 (10 points) Pure shear state of stress. Compute the state of strain in some axes of your choosing. Then find the principal strains and directions. Prove that in this case the principal directions of stress and strain coincide. Compute the maximum shear strains.

Solution: The state of stress in the given axes is:

$$[\sigma] = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}$$

Converting to material axes by using Mohr's circle



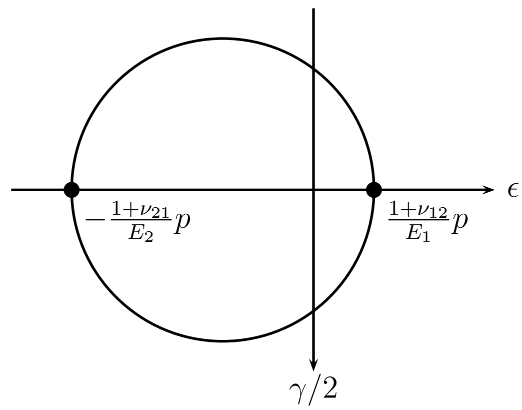
The transformed stress components are then

$$\tilde{\sigma}_{11} = p = \sigma_I$$
$$\tilde{\sigma}_{22} = -p = \sigma_{II}$$
$$\tilde{\sigma}_{12} = 0$$

We can now use the constitutive law, as the stress components are in the direction aligned with the fibers:

$$\epsilon_{11} = \frac{1}{E_1}\left(p - \nu_{12}(-p)\right) = \frac{1+\nu_{12}}{E_1}p = \epsilon_I$$
$$\epsilon_{22} = \frac{1}{E_2}\left((-p) - \nu_{21}p\right) = -\frac{1+\nu_{21}}{E_2}p = \epsilon_{II}$$
$$2\epsilon_{12} = \frac{0}{G_{12}}$$
$$[\epsilon] = \begin{pmatrix} \frac{1+\nu_{12}}{E_1}p & 0 \\ 0 & -\frac{1+\nu_{21}}{E_2}p \end{pmatrix}$$

Those are then principal directions of strain as well, then. Mohr's circle for strain in this case is:



The maximum shear strains are twice the radius of the circle:

$$R = |\epsilon_I - \epsilon_{II}| = \frac{1+\nu_{12}}{E_1}p - (-1)\frac{1+\nu_{21}}{E_2}p = \left(\frac{1+\nu_{12}}{E_1} + \frac{1+\nu_{21}}{E_2}\right)|p|$$

Question 2 [20 points]
A bar of length L, area A, mass density ρ, elastic modulus E and CTE α rotates around $\mathbf{e_2}$ with an angular velocity ω, as shown in Figure **??**. The free end $x = L$ is constrained from extensional motion by a ring housing (i.e. the displacement $u(L) = 0$ at the extremity A). After testing a first design, it was found that friction between the bar end and the housing produced unacceptable heating of the material surfaces. A solution that eliminates frictional forces at the interface completely is sought by modifying the temperature along the whole length of the bar uniformly by a value $\Delta\theta$.

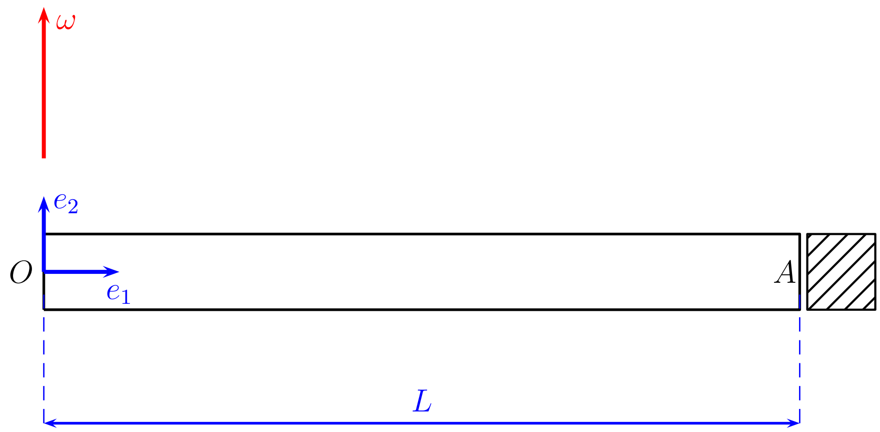


Figure 1: Rotating bar

2.1 (5 points) Specialize the governing equations of rod theory to this problem. What principles do they represent?

Solution: The equilibrium equation is:

$$N' + p = 0 \tag{1}$$

where p is the centripetal force acting on the bar $p = \rho\omega^2 x\, A$.
The equation enforcing compatibility and constitutive law is:

$$N = EA(u' - \alpha\Delta\theta) \tag{2}$$

2.2 (5 points) State the boundary conditions of this problem. Is the problem statically determinate or indeterminate? Justify your answer.

Solution: The boundary conditions are $u(0) = 0$ and $u(L) = 0$(zero displacement at both ends of the bar). The problem is statically indeterminate since neither boundary condition provides a value for the internal axial force $N(x)$ at any point of the domain.

2.3 (5 points) Reduce the system of equations above and integrate the resulting equation(s). Apply the boundary conditions to obtain the distribution of displacements $u(x)$ and internal axial load $N(x)$ as a function of the external actions ω and $\Delta\theta$.

Solution: Using (2) in (1) we obtain

$$(EA(u' - \alpha\Delta\theta))' + \rho\omega^2 Ax = 0 \tag{3}$$

Since $\Delta\theta$ is constant, the equation simplifies to:

$$u'' + \frac{\rho\omega^2}{E}x = 0 \tag{4}$$

Integrating twice we get

$$u(x) = -\frac{\rho\omega^2 x^3}{6E} + Bx + C \tag{5}$$

where B and C are arbitrary constants determined from the boundary conditions ($u(0) = 0$ and $u(L) = 0$) $B = \frac{\rho\omega^2}{E}\frac{L^2}{6}$ and $C = 0$. Now the displacement can be written in the following explicit form:

$$\boxed{u(x) = -\frac{\rho\omega^2 x^3}{6E} + \frac{\rho\omega^2 L^2 x}{6E} = \frac{\rho\omega^2 x\left(L^2 - x^2\right)}{6E}} \tag{6}$$

The normal force follows from (2), where

$$u'(x) = -\frac{\rho\omega^2 x^2}{2} + \frac{\rho\omega^2 L^2}{6} = \frac{\rho\omega^2\left(L^2 - 3x^2\right)}{6E}$$
$$\boxed{N(x) = EA\left[\frac{\rho\omega^2\left(L^2 - 3x^2\right)}{6E} - \alpha\Delta\theta\right]} \tag{7}$$

2.4 (5 points) Use the solution above to obtain an expression for the value of the temperature change $\Delta\theta$ which would make the normal $N(L)$, and therefore the friction forces vanish. Interpret the result in terms of the sign of $N(L)$ when there is no temperature change and the sign of the required temperature change to eliminate friction.

Solution:

$$N(L) = EA\left[\frac{\rho\omega^2\left(L^2 - 3L^2\right)}{6E} - \alpha\Delta\theta\right] = EA\left[-\frac{\rho\omega^2 L^2}{3E} - \alpha\Delta\theta\right] = 0, \rightarrow$$
$$\boxed{\Delta\theta = -\frac{\rho\omega^2 L^2}{3\alpha E}} \tag{8}$$

2.5 There are many additional interesting questions about this problem, but you are not asked to answer them:

- Is the displacement affected by the temperature applied in this case (why?)
- For the $\Delta\theta$ obtained, is there any penalty to be paid, e.g. in terms of the increase of $N(0)$? How much is it?
- Use realistic values of the problem parameters to get a sense of how practicable this approach of controlling mechanical response with temperature changes due to cooling or heating would be. For the requisite temperature change (cooling or heating?), one could do a thermodynamic calculation (wait for the Spring), to get a sense of the power consumption of such an approach.

Solution: DO NOT ANSWER THIS