16.001 - Materials & Structures
Problem Set #9

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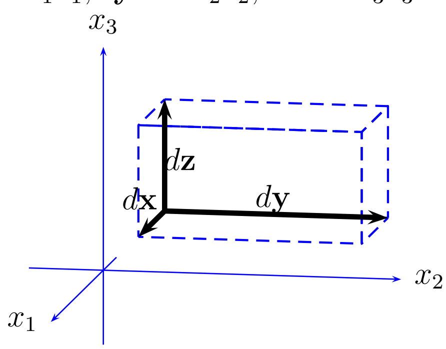
Question	Points
1	0
2	0
3	0
4	0
5	6
6	4
Total:	10

○ **Problems M-9.1** [0 points]

Analysis of volumetric deformations: In this problem we will use the tools we learned in vector calculus (e.g. 18.02) and our recently acquired knowledge of the strain tensor to analyze volumetric changes during the deformation of the material and how they relate to the stress tensor.

1.1 (1 point) Start by considering an infinitesimal prismatic volume element aligned with the cartesian axes whose sides are the vectors: $d\mathbf{x} = dx_1\mathbf{e}_1, d\mathbf{y} = dx_2\mathbf{e}_2, d\mathbf{z} = dx_3\mathbf{e}_3$. Show that its volume is given by the triple or mixed product: $dV = (d\mathbf{x} \times d\mathbf{y}) \cdot d\mathbf{z}$

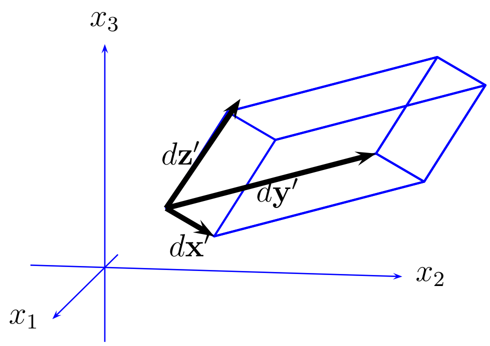
Solution: Consider the volume element defined by the differentials: $d\mathbf{x} = dx_1\mathbf{e}_1, d\mathbf{y} = dx_2\mathbf{e}_2, d\mathbf{z} = dx_3\mathbf{e}_3$ as shown in the figure:



From the figure and the definition of cross product, we see that $d\mathbf{x} \times d\mathbf{y} = (dx_1\mathbf{e}_1) \times (dx_2\mathbf{e}_2) = dx_1 dx_2 \mathbf{e}_3$, which is a vector pointing in direction $\mathbf{e}_3$ of magnitude equal to the area of the base of the volume element. Then: $(d\mathbf{x} \times d\mathbf{y}) \cdot d\mathbf{z} = (dx_1 dx_2 \mathbf{e}_3) \cdot dx_3\mathbf{e}_3 = dx_1 dx_2 dx_3 = V$, as sought.

1.2 (1 point)As discussed in class, the deformation is described by a vector field $\mathbf{x}' = \boldsymbol{\phi}(\mathbf{x})$ (also known as a deformation mapping), which assigns each material point at initial position $\mathbf{x}$ to the deformed position $\mathbf{x}'$. The undeformed differential vectors are mapped to: $d\mathbf{x}' = \boldsymbol{\nabla\phi} \cdot d\mathbf{x}, d\mathbf{y}' = \boldsymbol{\nabla\phi} \cdot d\mathbf{y}, d\mathbf{z}' = \boldsymbol{\nabla\phi} \cdot d\mathbf{z}$, where $\boldsymbol{\nabla\phi} = \frac{\partial \phi_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j$ is the gradient of the deformation. Give a geometric argument (use a sketch to support it) to show that the deformed volume dV' of the element dV is given by:

$$dV' = (d\mathbf{x}' \times d\mathbf{y}') \cdot d\mathbf{z}'$$



Solution: As before, the cross product will give a vector orthogonal to $d\mathbf{x}'$ $d\mathbf{y}'$ whose magnitude is the area of one of the bases of the deformed hexahedron. When dotted with $d\mathbf{z}'$ we obtain a scalar which is this area multiplied by the projection of the length of $d\mathbf{z}'$ on the plane normal, i.e. the height of the hexahedron. This gives the volume V', as sought.

1.3 (1 point) Replace the expressions of the deformed differentials to show that $\boxed{dV' = JdV}$, where

$$J = \left| \frac{\partial \phi_i}{\partial x_j} \right|$$

is the determinant of the matrix of partial derivatives of $\boldsymbol{\phi}$. This is the same expression you derived in vector calculus for the change of volume element when changing coordinate systems (e.g. from cartesian to spherical, etc.).

Solution:

$$d\mathbf{x}' = \boldsymbol{\nabla\phi} \cdot d\mathbf{x} = \left(\frac{\partial \phi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \right) \cdot (dx_1 \mathbf{e}_1) = \frac{\partial \phi_i}{\partial x_1} dx_1 \mathbf{e}_i$$

$$d\mathbf{y}' = \boldsymbol{\nabla\phi} \cdot d\mathbf{y} = \left(\frac{\partial \phi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \right) \cdot (dx_2 \mathbf{e}_2) = \frac{\partial \phi_i}{\partial x_2} dx_2 \mathbf{e}_i$$

$$d\mathbf{z}' = \boldsymbol{\nabla\phi} \cdot d\mathbf{z} = \left(\frac{\partial \phi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \right) \cdot (dx_3 \mathbf{e}_3) = \frac{\partial \phi_i}{\partial x_3} dx_3 \mathbf{e}_i$$

$$dV' = (d\mathbf{x}' \times d\mathbf{y}') \cdot d\mathbf{z}' = \left[\left(\frac{\partial \phi_i}{\partial x_1} dx_1 \mathbf{e}_i \right) \times \left(\frac{\partial \phi_j}{\partial x_2} dx_2 \mathbf{e}_j \right) \right] \cdot \left(\frac{\partial \phi_k}{\partial x_3} dx_3 \mathbf{e}_k \right)$$

$$= \left(\frac{\partial \phi_i}{\partial x_1} \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_k}{\partial x_3} dx_1 dx_2 dx_3 \right) \underbrace{(\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k}_{\epsilon_{ijk}}$$

$$= \left(\frac{\partial \phi_i}{\partial x_1} \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_k}{\partial x_3} \epsilon_{ijk} \right) \underbrace{dx_1 dx_2 dx_3}_{dV}$$

where we have used the permutation tensor and its properties studied earlier in the class. In particular, recalling that for a 3x3 matrix $|A| = a_{1i} a_{2j} a_{3k} \epsilon_{ijk}$, the expression in parenthesis is exactly the determinant of the matrix of components of $\boldsymbol{\nabla\phi}$, as sought.

1.4 (1 point) Now express the deformation mapping as the sum of the undeformed position $\mathbf{x}$ plus the displacement vector field $\mathbf{u}(\mathbf{x})$, as done in class, and express the volume change J in terms of the partial derivatives of $\mathbf{u}(\mathbf{x})$. Write the determinant in matrix form but do not expand it.

> **Solution:** from $\boldsymbol{\phi} = \mathbf{x} + \mathbf{u}$, we get:
>
> $$\frac{\partial \phi_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$
>
> $$J = \frac{\partial \phi_i}{\partial x_j} = \begin{matrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{matrix}$$

1.5 (1 point) Consider the case of small gradients $\boldsymbol{\nabla u} \ll 1$. By looking at your determinant, make and argument for the only surviving first-order terms (i.e. those not contaning products of partial derivatives) without expanding the determinant, to show that this linearized version of J, and therefore $\frac{V'}{V}$, is $1 + \nabla \cdot \mathbf{u}$ (note that this is one plus the divergence of $\mathbf{u}$, not the gradient. Then, easily show that $\frac{\Delta V}{V} = \frac{V'-V}{V} = \nabla \cdot \mathbf{u}$ and that this corresponds to the trace of the strain tensor ε_{kk}. We have thus identified that the volumetric deformation $\varepsilon_v = \varepsilon_{kk} = \frac{\partial u_i}{\partial x_i}$

> **Solution:** It is clear from the expression of the determinant that upon its expansion the only terms that will not involve products of the gradient components will be:
>
> $$J^{(\text{linearized})} = \left(\tfrac{V'}{V}\right)^{(\text{linearized})} = 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 1 + \frac{\partial u_k}{\partial x_k} = 1 + \nabla \cdot \mathbf{u}$$
>
> $$\frac{\Delta V}{V} = \frac{V' - V}{V} = \frac{V'}{V} - 1 = \nabla \cdot \mathbf{u}$$

1.6 (1 point) Use the constitutive relations for an isotropic linear elastic material in compliance form derived in class (Hooke's Law) to prove that:

$$\frac{\Delta V}{V} = \frac{\sigma_{kk}(1 - 2\nu)}{E} \tag{1}$$

From here, define the *bulk modulus* K as the ratio between the hydrostatic pressure *hydrostatic pressure* $p = \frac{\sigma_{kk}}{3}$ and the volumetric strain $\theta = \varepsilon_{kk}$, i.e.:

$$p = K\theta$$

Prove that:

$$\boxed{K = \frac{E}{3(1 - 2\nu)}}$$

Solution: Add the expressions of the normal strains in terms of the stresses for an isotropic linear elastic material:

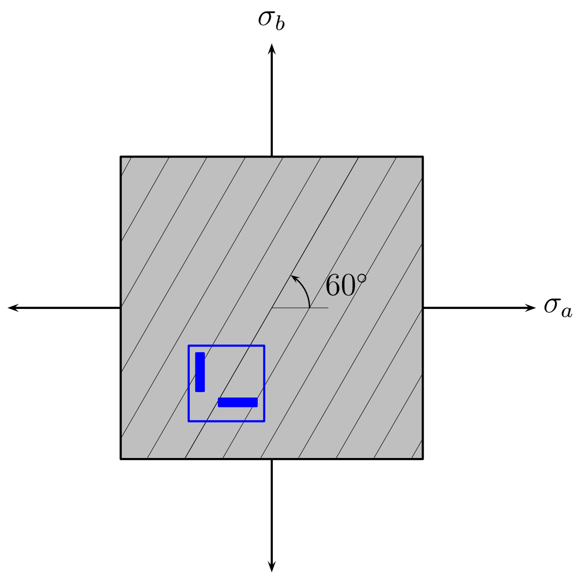
$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$
$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right]$$
$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

$$\underbrace{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}_{\varepsilon_{kk}=\theta} = \frac{1}{E}\left[(\sigma_{11} + \sigma_{22} + \sigma_{33}) - \nu\left((\sigma_{22} + \sigma_{33}) + (\sigma_{11} + \sigma_{33}) + (\sigma_{11} + \sigma_{22})\right)\right]$$
$$= \frac{1}{E}\left[(\sigma_{11} + \sigma_{22} + \sigma_{33}) - 2\nu\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right]$$
$$= \frac{1}{E}\underbrace{(\sigma_{11} + \sigma_{22} + \sigma_{33})}_{\sigma_{kk}=3p}(1 - 2\nu)$$
$$\theta = \underbrace{\frac{3(1-2\nu)}{E}}_{1/K} p$$

1.7 (1 point) What happens when $\nu \to 0.5$? What can you say about the material behavior in that limit?

Solution: Clearly, when $\nu \to 0.5$, the bulk modulus goes to ∞. This means that the material becomes infinitely stiff to volumetric deformations. This means that no matter how large the pressure is, the volume does not change (either in compression or dilatation).

○ **Problems M-9.2** [0 points]
An orthotropic composite material is loaded in the plane with tensile stresses σ_a and σ_b. The fibers in this composite are at a 60° angle to the $\mathbf{e}_1$ axis. Additionally, two strain gauges aligned with the $\mathbf{e}_1$ (labeled a) and $\mathbf{e}_2$ (labeled b) directions are placed in the material.



2.1 (3 points) Suppose you previously measured the Young's Modulus along the fiber direction to be $E_1 = 200$ GPa. Determine the remaining in-plane elastic constants describing the behavior of this material if the applied stresses are

$$\sigma_a = 100 \text{ MPa} \qquad \sigma_b = 50 \text{ MPa}$$

and the measured strains (from the strain gauges) and the shear strain in the $\mathbf{e}_1$-$\mathbf{e}_2$ plane are

$$\epsilon_a = 14 \times 10^{-4} \quad \epsilon_b = 4 \times 10^{-4} \quad \epsilon_{12} = -5.6 \times 10^{-4}$$

Hint: Remember the fact that the constitutive laws for orthotropic materials discussed in lecture apply only in material principal axes, i.e. they can only be used if the state of stress is described in directions aligned and perpendicular to the fibers.

Solution: Begin by transforming the stress and strain components to be aligned with the direction of the fibers. Applying the stress transformation relations with $\sigma_{11} = \sigma_a = 100$ MPa, $\sigma_{22} = \sigma_b = 50$ MPa, $\sigma_{12} = 0$:

$$\begin{aligned}
\tilde{\sigma}_{11} &= \frac{\sigma_a + \sigma_b}{2} + \frac{\sigma_a - \sigma_b}{2}\cos(120^\circ) = 62.5 \text{ MPa} \\
\tilde{\sigma}_{22} &= \frac{\sigma_a + \sigma_b}{2} - \frac{\sigma_a - \sigma_b}{2}\cos(120^\circ) = 87.5 \text{ MPa} \\
\tilde{\sigma}_{12} &= -\frac{\sigma_a - \sigma_b}{2}\sin(120^\circ) = -21.65 \text{ MPa}
\end{aligned}$$

We may transform the strains in the same manner, recognizing that $\epsilon_{11} = \epsilon_a = 14 \times 10^{-4}$ and $\epsilon_{22} = \epsilon_a = 4 \times 10^{-4}$. We also have $\epsilon_{12} = -5.6 \times\ 10^{-4}$. Thus, the transformation gives:

$$\tilde{\epsilon_{11}} = \frac{\epsilon_a + \epsilon_b}{2} + \frac{\epsilon_a - \epsilon_b}{2} \cos(120°) + \epsilon_{12} \sin(120°) = 1.65 \times 10^{-4}$$
$$\tilde{\epsilon_{22}} = \frac{\epsilon_a + \epsilon_b}{2} - \frac{\epsilon_a - \epsilon_b}{2} \cos(120°) - \epsilon_{12} \sin(120°) = 16.35 \times 10^{-4}$$
$$\tilde{\epsilon_{12}} = -\frac{\epsilon_a - \epsilon_b}{2} \sin(120°) + \epsilon_{12} \cos(120°) = -1.53 \times 10^{-4}$$

Given the stress and strain components in the direction aligned with the fibers, apply the constitutive equations

$$\epsilon_{11} = \frac{1}{E_1} \left(\sigma_{11} - \nu_{12}\sigma_{22}\right)$$
$$\epsilon_{22} = \frac{1}{E_2} \left(\sigma_{22} - \nu_{21}\sigma_{11}\right)$$
$$2\epsilon_{12} = \frac{\sigma_{12}}{G_{12}}$$

to determine the elastic constants. We are given $\boxed{E_1 = 200 \text{ GPa}}$. Apply the first stress-strain relation to solve for ν_{12}:

$$\epsilon_{11} = \frac{1}{E_1} \left(\sigma_{11} - \nu_{12}\sigma_{22}\right)$$
$$\rightarrow \nu_{12} = \frac{\sigma_{11} - E_1\epsilon_{11}}{\sigma_{22}} \rightarrow \boxed{\nu_{12} = 0.337}$$

Apply the second stress-strain relation and the reciprocity relation to solve for E_2:

$$\epsilon_{22} = \frac{\sigma_{22}}{E_2} - \frac{\nu_{21}}{E_2}\sigma_{11} = \frac{\sigma_{22}}{E_2} - \frac{\nu_{12}}{E_1}\sigma_{11}$$
$$\rightarrow E_2 = \frac{\sigma_{22}}{\epsilon_{22} + \frac{\nu_{12}}{E_1}\sigma_{11}} \rightarrow \boxed{E_2 = 50.28 \text{ GPa}}$$

Apply reciprocity to solve for ν_{21}

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$
$$\rightarrow \nu_{21} = \frac{E_2}{E_1}\nu_{12} \rightarrow \boxed{\nu_{21} = 0.0847}$$

Finally, obtain G_{12} from the third stress strain relation

$$2\epsilon_{12} = \frac{\sigma_{12}}{G_{12}} \rightarrow G_{12} = \frac{\sigma_{12}}{2\epsilon_{12}} \rightarrow \boxed{G_{12} = 70.75 \text{ GPa}}$$

Thus, we find the following constants:

$$E_1 = 200 \text{ GPa}$$
$$E_2 = 50.28 \text{ GPa}$$
$$\nu_{12} = 0.337$$
$$\nu_{21} = 0.0847$$
$$G_{12} = 70.75 \text{ GPa}$$

2.2 (3 points) One failure mechanism of such composite materials is the buckling of fibers due to compressive stresses. Since the orthotropic constitutive relations relate stresses and strains in the composite material, one should be able to determine whether a fiber experiences a tensile or a compressive stress along its direction based on strains. Derive a condition that the strain gauge readings ϵ_a and ϵ_b, and the shear strain ϵ_{12}, should satisfy so that the stress along the fibers is not compressive. Your condition should include material properties as well. *Note: Don't use any of the numeric values for the elastic constants found in part (a), or assume the stress/strain components from the previous part. Solve symbolically in terms of ϵ_a, ϵ_b, ϵ_{12} and the material properties.*

Solution: Our goal is to ensure that the axial stress along the direction of the fibers $\tilde{\sigma}_{11}$ is positive, and represent this in terms of strains. We begin with the first two stress-strain relations in the direction of the fibers:

$$\tilde{\epsilon}_{11} = \frac{1}{E_1}\tilde{\sigma}_{11} - \frac{\nu_{12}}{E_1}\tilde{\sigma}_{22}$$
$$\tilde{\epsilon}_{22} = \frac{1}{E_2}\tilde{\sigma}_{22} - \frac{\nu_{21}}{E_2}\tilde{\sigma}_{11}$$

Solve for σ_{11} by eliminating σ_{22}. This can be done by multiplying the second equation by ν_{21}

$$\tilde{\epsilon}_{11} = \frac{1}{E_1}\tilde{\sigma}_{11} - \frac{\nu_{12}}{E_1}\tilde{\sigma}_{22}$$
$$\nu_{21}\tilde{\epsilon}_{22} = \frac{\nu_{21}}{E_2}\tilde{\sigma}_{22} - \frac{\nu_{21}^2}{E_2}\tilde{\sigma}_{11}$$

and adding the two equations together (note the reciprocity relation $\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$ causes terms to cancel). We have:

$$\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22} = \left(\frac{1}{E_1} - \frac{\nu_{21}^2}{E_2}\right)\tilde{\sigma}_{11}$$

Solving for $\tilde{\sigma}_{11}$:

$$\tilde{\sigma}_{11} = \frac{\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22}}{\left(\frac{1}{E_1} - \frac{\nu_{21}^2}{E_2}\right)}$$

Applying the reciprocity relation and multiplying the numerator and denominator by E_1 gives the following form for $\tilde{\sigma}_{11}$:

$$\tilde{\sigma}_{11} = \frac{E_1(\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22})}{1 - \nu_{12}\nu_{21}}$$

The $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_{22}$, which are the strains in the direction of the fibers, may be written in terms of $\epsilon_{11} = \epsilon_a$, $\epsilon_{22} = \epsilon_b$, and ϵ_{12} using the transformation relations:

$$\begin{aligned}
\tilde{\epsilon}_{11} &= \frac{\epsilon_a + \epsilon_b}{2} + \frac{\epsilon_a - \epsilon_b}{2}\cos(120°) + \epsilon_{12}\sin(120°) \\
&= \frac{\epsilon_a + \epsilon_b}{2} + \frac{\epsilon_a - \epsilon_b}{2}\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\epsilon_{12} \\
&= \frac{\epsilon_a}{4} + \frac{3\epsilon_b}{4} + \frac{\sqrt{3}}{2}\epsilon_{12}
\end{aligned}$$

$$\begin{aligned}
\tilde{\epsilon}_{22} &= \frac{\epsilon_a + \epsilon_b}{2} - \frac{\epsilon_a - \epsilon_b}{2}\cos(120°) - \epsilon_{12}\sin(120°) \\
= \tilde{\epsilon}_{22} &= \frac{\epsilon_a + \epsilon_b}{2} - \frac{\epsilon_a - \epsilon_b}{2}\left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{2}\epsilon_{12} \\
&= \frac{3\epsilon_a}{4} + \frac{\epsilon_b}{4} - \frac{\sqrt{3}}{2}\epsilon_{12}
\end{aligned}$$

Thus, to prevent compression of the fibers, we must have

$$\boxed{\frac{E_1(\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22})}{1 - \nu_{12}\nu_{21}} > 0}$$

where

$$\boxed{\tilde{\epsilon}_{11} = \frac{\epsilon_a}{4} + \frac{3\epsilon_b}{4} + \frac{\sqrt{3}}{2}\epsilon_{12}}$$

$$\boxed{\tilde{\epsilon}_{22} = \frac{3\epsilon_a}{4} + \frac{\epsilon_b}{4} - \frac{\sqrt{3}}{2}\epsilon_{12}}$$

○ **Problems M-9.3** [0 points]
A composite material is subjected to the state of plane stress shown in Figure 1. The composite is a polymer matrix reinforced with unidirectional fibers that are aligned at α from horizontal. The Young's moduli in the directions parallel and perpendicular to the fiber are referred to as E_1 and E_2.

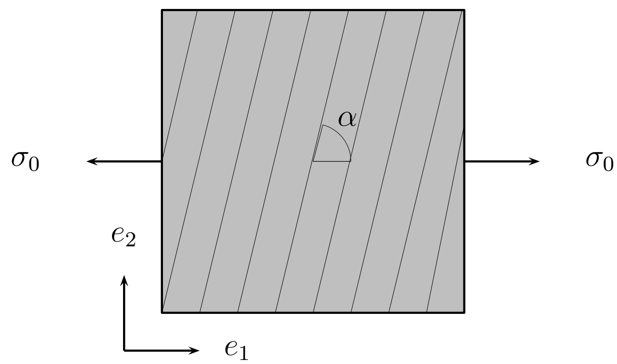


Figure 1: Composite Materials State of Stress

3.1 (1 point) Find the strain components in the basis (e_1, e_2), with the following numerical values:

$$E_1 = 180 \text{ GPa} \qquad E_2 = 70 \text{ GPa} \qquad \nu_{12} = 0.35$$

$$G = 90 \text{ GPa} \qquad \alpha = 75°$$

Solution:
For this first part, the procedure follows three main steps: we rotate the stresses in a direction aligned with the fibers, then we use the constitutive relations for orthotropic materials to get the strains (still in a direction oriented with the fibers), and finally we rotate back the material to the initial basis.
The stress state with respect to (e_1, e_2) is the folllowing:

$$\sigma_{11} = \sigma_0$$
$$\sigma_{22} = 0$$
$$\sigma_{12} = 0$$

To find the axial stresses in the direction parallel and perpendicular to the fibers, we need to perform a stress transformation on the stress state. We use the following equations with a positive α since the rotation is counter-clockwise:

$$\tilde{\sigma}_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} cos(2\alpha) + \sigma_{12} sin(2\alpha)$$
$$\tilde{\sigma}_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} cos(2\alpha) - \sigma_{12} sin(2\alpha)$$
$$\tilde{\sigma}_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2} sin(2\alpha) + \sigma_{12} cos(2\alpha)$$

The numerical values of the stresses in the new basis aligned with the fibers is the following:

$$\tilde{\sigma}_{11} = 0.067\ \sigma_0$$
$$\tilde{\sigma}_{22} = 0.93\ \sigma_0$$
$$\tilde{\sigma}_{12} = -0.25\ \sigma_0$$

Then we apply the constitutive relations for orthotropic materials. The equations are the following:

$$\tilde{\epsilon}_{11} = \frac{1}{E_1}(\tilde{\sigma}_{11} - \nu_{12}\tilde{\sigma}_{22})$$
$$\tilde{\epsilon}_{22} = \frac{1}{E_2}(\tilde{\sigma}_{22} - \nu_{21}\tilde{\sigma}_{11})$$
$$\tilde{\epsilon}_{12} = \frac{\tilde{\sigma}_{12}}{2G}$$

We recall that the reciprocity relation gives the following relation:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

Therefore, we find the value $\nu_{21} = 0.136$, and the numerical results for the strains in the direction of the fibers are:

$$\tilde{\epsilon}_{11} = -1.43 \times 10^{-12}\sigma_0$$
$$\tilde{\epsilon}_{22} = 1.31 \times 10^{-11}\sigma_0$$
$$\tilde{\epsilon}_{12} = -1.39 \times 10^{-12}\sigma_0$$

Finally, the rotation back to the initial basis (e_1, e_2) with an angle -α gives the following values for the strains:

$$\epsilon_{11} = -1.29 \times 10^{-11}\sigma_0$$
$$\epsilon_{22} = -1.15 \times 10^{-12}\sigma_0$$
$$\epsilon_{12} = -2.46 \times 10^{-12}\sigma_0$$

3.2 (1 point) Interpret your results and explain how you think the material will deform.

Solution: We observe that the in the basis (e_1, e_2), there is some shear strain, even though there was no shear stress in the same basis. Since the strain is negative, the material is likely to deform in a similar way as shown in Figure 2, the blue color indicating the deformed configuration. This can also be explained physically by the fact that $E_1 > E_2$, and therefore the material extends less in the direction parallel to the fiber than in the direction perpendicular to the fibers.

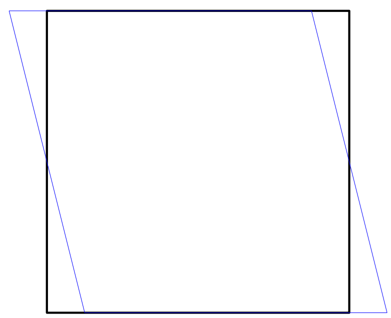


Figure 2: Composite Materials State of Stress

3.3 (1 point) Find the principal directions of ϵ and σ. Do they coincide? Do they usually coincide in the case of orthotropic materials? What about isotropic materials?

Solution: In the basis (e_1, e_2), there is no shear stress, so this is the principal direction for σ (in other words, $\alpha_p^\sigma = 0$). Concerning the strains, the principal direction are found by rotating the basis (e_1, e_2) with an angle α_p such that:

$$tan(2\alpha_p) = \frac{2\epsilon_{12}}{\epsilon_{11} - \epsilon_{22}}$$

$$\alpha_p^\epsilon = \boxed{-9.62°}$$

Since α_p^σ and α_p^ϵ are not equal, the principal directions do not coincide. They do not coincide in general in the case of orthotropic materials, so we should not be surprised by the results. On the other hand, we saw in class that the principal directions of ϵ and σ always coincide in the case of isotropic materials.

3.4 (1 point) If the principal directions of ϵ and σ do not coincide in this configuration, can you find a value of α for which they would coincide? (all other parameters being equal)

Solution: We are looking for the angles α at which we get $\alpha_p^\epsilon = 0$ (or at which $\epsilon_{12} = 0$). Using the same method as in the first part of the exercise without evaluating α at a particular value, we get:

$$\tilde{\sigma}_{11} = \frac{1}{2}(1 + cos(2\alpha))\sigma_0$$
$$\tilde{\sigma}_{22} = \frac{1}{2}(1 - cos(2\alpha))\sigma_0$$
$$\tilde{\sigma}_{12} = -\frac{1}{2}sin(2\alpha)\sigma_0$$

Then using the constitutive equations for isotropic materials, we get:

$$\tilde{\epsilon}_{11} = \frac{1}{E_1}(\tilde{\sigma}_{11} - \nu_{12}\tilde{\sigma}_{22}) = \frac{\sigma_0}{2E_1}\left[(1 - \nu_{12}) + cos(2\alpha)(1 + \nu_{12})\right]$$
$$\tilde{\epsilon}_{22} = \frac{1}{E_2}(\tilde{\sigma}_{22} - \nu_{21}\tilde{\sigma}_{11}) = \frac{\sigma_0}{2E_2}\left[(1 - \nu_{21}) - cos(2\alpha)(1 + \nu_{21})\right]$$
$$\tilde{\epsilon}_{12} = \frac{\tilde{\sigma}_{12}}{2G} = -\frac{\sigma_0}{2G}sin(2\alpha)$$

Finally, the last step is to rotate back the strain to the initial basis. We are only interested in the shear strain. Indeed, we are looking for the angles α at which ϵ_{12} vanish. The formula for the shear strain is:

$$\epsilon_{12}(\alpha) = -\frac{\tilde{\epsilon}_{11} - \tilde{\epsilon}_{22}}{2}sin(-2\alpha) + \tilde{\epsilon}_{12}cos(-2\alpha)$$

Pluging the values of $\tilde{\epsilon}_{11}$, $\tilde{\epsilon}_{22}$, and $\tilde{\epsilon}_{12}$ yield to a complex equation that one can not solve analytically. Using an external software (Mathematica, Matlab,etc), we can find the values of α which cancel $\epsilon_{12}(\alpha)$. By looking at the interval $[-90°; 90°]$, we find 5 values of α that fulfill this condition:

$$\boxed{\alpha = \pm 90°} \quad \boxed{\alpha = \pm 23.11°} \quad \boxed{\alpha = 0}$$

○ **Problems M-9.4** [0 points]
Consider the following orthotropic costitutive law for wood

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{Bmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{pmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}$$

4.1 (1 point) The wood is in pure compression in the σ_{11} direction and it is in purely compressive strain in this direction as well. What can we conclude from this regarding the parameters in the constitutive law?

Solution: The equations imply directly the following results

$$2\epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} = 0$$

$$2\epsilon_{33} = -\frac{\nu_{13}}{E_1}\sigma_{11} = 0$$

These imply that ν_{21}, ν_{31}, ν_{12} and ν_{13} are zero. They do not imply that all the poisson ratios are zero.

4.2 (1 point) Now consider the wood as being a part of the side of a tree. The $\mathbf{e}_1$ direction points up away from the ground and the $\mathbf{e}_2$ direction points radially into the tree. Assume that all the poisson ratios are not equal to zero and are strictly positive. Further assume that none of the three young's moduli are equal so that this is an orthotropic material. If you had the ability to select the three youngs moduli of the wood on the outside of this tree, which would you make largest and smallest in order to allow the tree trunk to hold the largest vertical loads with minimal straining? It might help to consider that the material on the outside of the tree needs to contain all the material inside the core just like a pressure vessel contains a gas.

Solution: In order to make the tree strongest with respect to loadings σ_{11}, E_1 should be chosen as the largest youngs modulus. This will prevent the tree from compressing vertically under vertical loading. Because the poisson's ratio's are all selected to be positive this compressive stress will produce components of positive strain in the ϵ_{22} and ϵ_{33} components. We can imagine that at the center of this tree where the idea of a radial direction is not well defined that the wood in the center of the horizontal plane of the trunk would be in a state of uniform and positive strain in both directions. In order to contain the pressure induced by the poisson effect and the vertical loading, a large youngs modulus in the hoop direction that is not the radial direction is prefered. The smallest youngs modulus would be in the radial direction. Many woods have these material properties.

4.3 (1 point) If you cut down this tree you can measure the youngs modulus E_1 by placing the wood in pure tension oriented along the vertical direction. You put the material under 10 MPa of stress along the grain and observe 800 microstrain along the grain. None of the other strain gages read any strain so you conclude that your test was in the principle axis of the material. Find the largest youngs modulus of the wood.

Solution: The youngs modulus is simply the stress divided by the strain which yields a modulus of 12.5 GPa.

4.4 (1 point) Your test to find the other two youngs moduli does not go as smoothly. You take a piece of material that is thin in the $\mathbf{e}_1$ direction and you place it under 10 MPa of uniaxial tension in some x direction. The tree is not a perfect circle and you have trouble orienting your loading in the direction that is the radial direction. Consequently, your strain gages find that the material strained 1000 microstrain in the x direction and -800 microstrain in the other y direction. It also strained 50 microstrain in the direction 45 degrees off from the x axis and the y axis. Compute the principal strains in your test and the angle of rotation needed to attain a frame of principle strain. Are the principle stresses and strains in the same frame? Comment on if you can find the other two moduli.

Solution: Using the shear strain transformation equation we can compute the angle of rotation needed to get this state of strain into the state of principle strain

$$0 = -\frac{1000 - (-800)}{2}\sin(2\theta) + 50\cos(2\theta)$$
$$\frac{50}{900} = \tan(2\theta)$$
$$\theta = 1.5899$$

From this we can find that the principle strains are 1000.97 and -800.97 microstrain so we are very close to a state of principle strain. The frame of principle stress and strain do not need to line up if the constitive law is not isotropic. In this case, the wood has a prefered direction that it likes to deform in, and if you accidentally stress the wood at some slight angle to this direction then the wood will strain slightly more in the components that areclose to the direction in which it is less stiff.

The state of stress in this axis is three values as is the state of strain. So this test gives us 3 pieces of information regarding our constitutive law. For orthotropic

2D materials we have 5 parameters and one constraint from the poisson ratio terms being equal. This is four independent unknowns plus the unknown angle of the principle axis of the material. So we need to get two more independent pieces of information about this material to really understand its constitutive law.

4.5 (1 point) Another student takes a sample of identical material and cuts it at some strange angle and proceeds with a much more complicated calculation of the kind that you just did. They compute that the youngs moduli in the plane of the wood that they have cut are 9 and 15 GPa. Is this possible given your results?

Solution: No they cannot find a youngs modulus larger then the largest principle moduli by cutting in a different plane. The principle axis are the stifffest and least stiff directions in the material, so there is no stiffer direction.

4.6 (1 point) You now take your sample and perform a different uniaxial tension test in the y direction again using 10 MPa. You measure the following strains: 900 micro strain in the y and -850 in the x. You also see -300 microstrain of shear. You are concerned these results are incorrect since the shear strain is so large and you worry this might have caused the test to be performed with some nonzero shear stress. Is it possibly to identify if these measurements are consistent?

Solution: Simply state the constitutive law in the rotated frame that is needed to get to the materials axis

$$\begin{Bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{Bmatrix} = \begin{pmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{2G_{12}}\end{pmatrix}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}100 + 900\cos 2\theta + 50\sin\theta\\ 100 - 900\cos 2\theta - 50\sin\theta\\ -900\sin 2\theta + 50\cos\theta\end{Bmatrix} = \begin{pmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{2G_{12}}\end{pmatrix}\begin{Bmatrix}\frac{10}{2}(1+\cos 2\theta)\\ \frac{10}{2}(1-\cos 2\theta)\\ -\frac{10}{2}(\sin 2\theta)\end{Bmatrix}$$

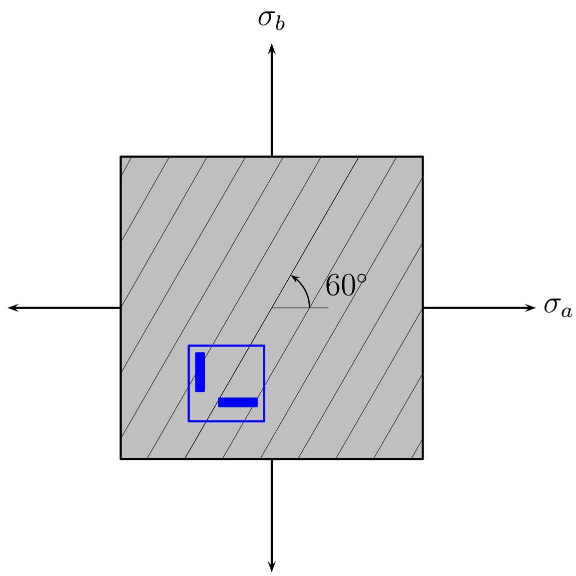
Then for the second tension test

$$\begin{Bmatrix}25 - 875\cos 2\theta - 300\sin\theta\\ 25 + 875\cos 2\theta + 300\sin\theta\\ 875\sin 2\theta + 300\cos\theta\end{Bmatrix} = \begin{pmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{2G_{12}}\end{pmatrix}\begin{Bmatrix}\frac{10}{2}(1-\cos 2\theta)\\ \frac{10}{2}(1+\cos 2\theta)\\ \frac{10}{2}(\sin 2\theta)\end{Bmatrix}$$

Here you state the constitutive law in some frame in terms of the angle θ that is needed to rotate to the material principle axis. We can solve this system to find E_1, E_2, ν_{12}, ν_{21}, G and θ. If we find that the following condition is not satisfied then we can conclude these measurements are inconsistent and that perhaps some shear stress was present during the test

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

○ **Problems M-9.5** [6 points]
An orthotropic composite material is loaded in the plane with tensile stresses σ_a and σ_b. The fibers in this composite are at a 60° angle to the $\mathbf{e}_1$ axis. Additionally, two strain gauges aligned with the $\mathbf{e}_1$ (labeled a) and $\mathbf{e}_2$ (labeled b) directions are placed in the material.



5.1 (3 points) Suppose you previously measured the Young's Modulus along the fiber direction to be $E_1 = 200$ GPa. Determine the remaining in-plane elastic constants describing the behavior of this material if the applied stresses are

$$\sigma_a = 100 \text{ MPa} \qquad \sigma_b = 50 \text{ MPa}$$

and the measured strains (from the strain gauges) and the shear strain in the $\mathbf{e}_1$-$\mathbf{e}_2$ plane are

$$\epsilon_a = 14 \times 10^{-4} \quad \epsilon_b = 4 \times 10^{-4} \quad \epsilon_{12} = -5.6 \times 10^{-4}$$

Hint: Remember the fact that the constitutive laws for orthotropic materials discussed in lecture apply only in material principal axes, i.e. they can only be used if the state of stress is described in directions aligned and perpendicular to the fibers.

Solution: Begin by transforming the stress and strain components to be aligned with the direction of the fibers. Applying the stress transformation relations with $\sigma_{11} = \sigma_a = 100$ MPa, $\sigma_{22} = \sigma_b = 50$ MPa, $\sigma_{12} = 0$:

$$\tilde{\sigma}_{11} = \frac{\sigma_a + \sigma_b}{2} + \frac{\sigma_a - \sigma_b}{2}\cos(120°) = 62.5 \text{ MPa}$$
$$\tilde{\sigma}_{22} = \frac{\sigma_a + \sigma_b}{2} - \frac{\sigma_a - \sigma_b}{2}\cos(120°) = 87.5 \text{ MPa}$$
$$\tilde{\sigma}_{12} = -\frac{\sigma_a - \sigma_b}{2}\sin(120°) = -21.65 \text{ MPa}$$

We may transform the strains in the same manner, recognizing that $\epsilon_{11} = \epsilon_a = 14 \times 10^{-4}$ and $\epsilon_{22} = \epsilon_a = 4 \times 10^{-4}$. We also have $\epsilon_{12} = -5.6 \times\ 10^{-4}$. Thus, the transformation gives:

$$\tilde{\epsilon_{11}} = \frac{\epsilon_a + \epsilon_b}{2} + \frac{\epsilon_a - \epsilon_b}{2}\cos(120°) + \epsilon_{12}\sin(120°) = 1.65 \times 10^{-4}$$
$$\tilde{\epsilon_{22}} = \frac{\epsilon_a + \epsilon_b}{2} - \frac{\epsilon_a - \epsilon_b}{2}\cos(120°) - \epsilon_{12}\sin(120°) = 16.35 \times 10^{-4}$$
$$\tilde{\epsilon_{12}} = -\frac{\epsilon_a - \epsilon_b}{2}\sin(120°) + \epsilon_{12}\cos(120°) = -1.53 \times 10^{-4}$$

Given the stress and strain components in the direction aligned with the fibers, apply the constitutive equations

$$\epsilon_{11} = \frac{1}{E_1}(\sigma_{11} - \nu_{12}\sigma_{22})$$
$$\epsilon_{22} = \frac{1}{E_2}(\sigma_{22} - \nu_{21}\sigma_{11})$$
$$2\epsilon_{12} = \frac{\sigma_{12}}{G_{12}}$$

to determine the elastic constants. We are given $\boxed{E_1 = 200 \text{ GPa}}$. Apply the first stress-strain relation to solve for ν_{12}:

$$\epsilon_{11} = \frac{1}{E_1}(\sigma_{11} - \nu_{12}\sigma_{22})$$
$$\rightarrow \nu_{12} = \frac{\sigma_{11} - E_1\epsilon_{11}}{\sigma_{22}} \rightarrow \boxed{\nu_{12} = 0.337}$$

Apply the second stress-strain relation and the reciprocity relation to solve for E_2:

$$\epsilon_{22} = \frac{\sigma_{22}}{E_2} - \frac{\nu_{21}}{E_2}\sigma_{11} = \frac{\sigma_{22}}{E_2} - \frac{\nu_{12}}{E_1}\sigma_{11}$$
$$\rightarrow E_2 = \frac{\sigma_{22}}{\epsilon_{22} + \frac{\nu_{12}}{E_1}\sigma_{11}} \rightarrow \boxed{E_2 = 50.28 \text{ GPa}}$$

Apply reciprocity to solve for ν_{21}

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$
$$\rightarrow \nu_{21} = \frac{E_2}{E_1}\nu_{12} \rightarrow \boxed{\nu_{21} = 0.0847}$$

Finally, obtain G_{12} from the third stress strain relation

$$2\epsilon_{12} = \frac{\sigma_{12}}{G_{12}} \rightarrow G_{12} = \frac{\sigma_{12}}{2\epsilon_{12}} \rightarrow \boxed{G_{12} = 70.75 \text{ GPa}}$$

Thus, we find the following constants:

$$\begin{aligned} E_1 &= 200 \text{ GPa} \\ E_2 &= 50.28 \text{ GPa} \\ \nu_{12} &= 0.337 \\ \nu_{21} &= 0.0847 \\ G_{12} &= 70.75 \text{ GPa} \end{aligned}$$

5.2 (3 points) One failure mechanism of such composite materials is the buckling of fibers due to compressive stresses. Since the orthotropic constitutive relations relate stresses and strains in the composite material, one should be able to determine whether a fiber experiences a tensile or a compressive stress along its direction based on strains. Derive a condition that the strain gauge readings ϵ_a and ϵ_b, and the shear strain ϵ_{12}, should satisfy so that the stress along the fibers is not compressive. Your condition should include material properties as well. *Note: Don't use any of the numeric values for the elastic constants found in part (a), or assume the stress/strain components from the previous part. Solve symbolically in terms of* ϵ_a, ϵ_b, ϵ_{12} *and the material properties.*

Solution: Our goal is to ensure that the axial stress along the direction of the fibers $\tilde{\sigma}_{11}$ is positive, and represent this in terms of strains. We begin with the first two stress-strain relations in the direction of the fibers:

$$\begin{aligned} \tilde{\epsilon}_{11} &= \frac{1}{E_1}\tilde{\sigma}_{11} - \frac{\nu_{12}}{E_1}\tilde{\sigma}_{22} \\ \tilde{\epsilon}_{22} &= \frac{1}{E_2}\tilde{\sigma}_{22} - \frac{\nu_{21}}{E_2}\tilde{\sigma}_{11} \end{aligned}$$

Solve for σ_{11} by eliminating σ_{22}. This can be done by multiplying the second equation by ν_{21}

$$\begin{aligned} \tilde{\epsilon}_{11} &= \frac{1}{E_1}\tilde{\sigma}_{11} - \frac{\nu_{12}}{E_1}\tilde{\sigma}_{22} \\ \nu_{21}\tilde{\epsilon}_{22} &= \frac{\nu_{21}}{E_2}\tilde{\sigma}_{22} - \frac{\nu_{21}^2}{E_2}\tilde{\sigma}_{11} \end{aligned}$$

and adding the two equations together (note the reciprocity relation $\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$ causes terms to cancel). We have:

$$\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22} = \left(\frac{1}{E_1} - \frac{\nu_{21}^2}{E_2}\right)\tilde{\sigma}_{11}$$

Solving for $\tilde{\sigma}_{11}$:

$$\tilde{\sigma}_{11} = \frac{\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22}}{\left(\frac{1}{E_1} - \frac{\nu_{21}^2}{E_2}\right)}$$

Applying the reciprocity relation and multiplying the numerator and denominator by E_1 gives the following form for $\tilde{\sigma}_{11}$:

$$\tilde{\sigma}_{11} = \frac{E_1(\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22})}{1 - \nu_{12}\nu_{21}}$$

The $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_{22}$, which are the strains in the direction of the fibers, may be written in terms of $\epsilon_{11} = \epsilon_a$, $\epsilon_{22} = \epsilon_b$, and ϵ_{12} using the transformation relations:

$$\begin{aligned}
\tilde{\epsilon}_{11} &= \frac{\epsilon_a + \epsilon_b}{2} + \frac{\epsilon_a - \epsilon_b}{2}\cos(120^\circ) + \epsilon_{12}\sin(120^\circ) \\
&= \frac{\epsilon_a + \epsilon_b}{2} + \frac{\epsilon_a - \epsilon_b}{2}\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\epsilon_{12} \\
&= \frac{\epsilon_a}{4} + \frac{3\epsilon_b}{4} + \frac{\sqrt{3}}{2}\epsilon_{12}
\end{aligned}$$

$$\begin{aligned}
\tilde{\epsilon}_{22} &= \frac{\epsilon_a + \epsilon_b}{2} - \frac{\epsilon_a - \epsilon_b}{2}\cos(120^\circ) - \epsilon_{12}\sin(120^\circ) \\
= \tilde{\epsilon}_{22} &= \frac{\epsilon_a + \epsilon_b}{2} - \frac{\epsilon_a - \epsilon_b}{2}\left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{2}\epsilon_{12} \\
&= \frac{3\epsilon_a}{4} + \frac{\epsilon_b}{4} - \frac{\sqrt{3}}{2}\epsilon_{12}
\end{aligned}$$

Thus, to prevent compression of the fibers, we must have

$$\boxed{\frac{E_1(\tilde{\epsilon}_{11} + \nu_{21}\tilde{\epsilon}_{22})}{1 - \nu_{12}\nu_{21}} > 0}$$

where

$$\boxed{\tilde{\epsilon}_{11} = \frac{\epsilon_a}{4} + \frac{3\epsilon_b}{4} + \frac{\sqrt{3}}{2}\epsilon_{12}}$$

$$\boxed{\tilde{\epsilon}_{22} = \frac{3\epsilon_a}{4} + \frac{\epsilon_b}{4} - \frac{\sqrt{3}}{2}\epsilon_{12}}$$

○ **Problems M-9.6** [4 points]
A composite propeller blade is being designed using fiber reinforced composites. The blade must be able to accelerate from a low RPM to a high RPM which induces a large shear stress on the fibers. Towards the tip the blade sweeps backwards and so the fibers are bent and do not run in the radial direction. In fact, the fiber angle off of the radial direction varies in the following manner as a function of the radial coordinate along the blade

$$\alpha = \alpha_1 R$$

Assume the shear loading on the blade varies in the following manner where the $\mathbf{e}_1$ direction is the radial coordinate

$$\sigma_{12} = (R_o - R)(\tau_0)$$
$$\sigma_{11} = \sigma_0(R_0^2 - R^2)$$

The constitutive law for this material in the fiber direction is given by

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{2G_{xy}} \end{pmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

6.1 (2 points) Write the equations to find the strain components in the frame of the radial coordinate system (Not the fiber frame). Explain how you would use these equations to compute the values of the strains.

Solution: We use the familiar stress transformation equations to compute the stress in the x-y coordinate frame of the fibers

$$\sigma_{xx} = \frac{\sigma_0(R_0^2 - R^2)}{2}(1 + \cos 2R\alpha_1) + (R_o - R)(\tau_0)\sin 2R\alpha$$
$$\sigma_{yy} = \frac{\sigma_0(R_0^2 - R^2)}{2}(1 - \cos 2R\alpha_1) - (R_o - R)(\tau_0)\sin 2R\alpha$$
$$\sigma_{xy} = -\frac{\sigma_0(R_0^2 - R^2)}{2}(\sin 2R\alpha_1) + (R_o - R)(\tau_0)\cos 2R\alpha$$

Thus we can state

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{2G_{xy}} \end{pmatrix} \begin{Bmatrix} \frac{\sigma_0(R_0^2-R^2)}{2}(1 + \cos 2R\alpha_1) + (R_o - R)(\tau_0)\sin 2R\alpha \\ \frac{\sigma_0(R_0^2-R^2)}{2}(1 - \cos 2R\alpha_1) - (R_o - R)(\tau_0)\sin 2R\alpha \\ -\frac{\sigma_0(R_0^2-R^2)}{2}(\sin 2R\alpha_1) + (R_o - R)(\tau_0)\cos 2R\alpha \end{Bmatrix}$$

Then we can use the following transformation to compute strain in the radial coordinate system

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{pmatrix} \frac{1+\cos(2R\alpha)}{2} & \frac{1-\cos(2R\alpha)}{2} & \sin(-2R\alpha) \\ \frac{1-\cos(2R\alpha)}{2} & \frac{1+\cos(2R\alpha)}{2} & -\sin(-2R\alpha) \\ -\frac{\sin(-2R\alpha)}{2} & \frac{\sin(-2R\alpha)}{2} & \cos(2R\alpha) \end{pmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix}$$

6.2 (2 points) Assuming the radial stress is much larger then the shear stress, where would you expect to see high strains (ϵ_{11} and ϵ_{12}) in the radial, theta coordinate system on the propeller blade? What parameters influence these values?

Solution: We would expect large strains in the radial direction at the root of the blade since the loading is largest in this direction. We might expect a similar effect due to the increase in shear loading, but if we consider that the shear loading is small we see that the stress in the radial direction stresses both the fibers and the direction perpendicular to them as the fibers curve. If the fibers are very very strong, then the other direction will strain far more and the shear strain induced by the axial force will be negative. If the fibers are not that strong then the axial force will actually contribute positive shear strain. Since the fibers are turning all the time as you proceed along the radial direction, the hydrostatic pressure is being shifted away from the fiber direction and added to the nonfiber direction. Thus I would expect that far out along the blade this effect is helpful, but close in where the fibers are potentially overloaded relative to the loading on the matrix, there will probably be large positive shearing.