16.001 - Materials & Structures
Problem Set #1

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Problem M-1.1
Breguet Range Equation

(a) The purpose of this exercise is to estimate the range of the Airbus 340-500 from the flight data provided in lecture. Based on the charts given (you may also find it useful to consult the official specifications available on the Airbus web site estimate each one of the parameters in the Breguet Range Equation, explaning the process by which you obtain each parameter. Then, use the Range Equation to estimate the maximum range. If you don't find information about the *overall propulsive efficiency* and *aerodynamic efficiency* you can use the following values: $\eta_0 = 0.35, L/D = 17$. Compare your estimate with the information available online.

Solution: From the charts for the JFK to Abu Dhabi flight and the information on the Airbus web site, we can obtain or estimate the following parameters:

$$L = 380\text{t}, \quad D = 23\text{t}, \quad \eta_{\text{overall}} = 0.35, \quad h_f = 42\text{M J/kg}, \quad \text{MTOW} = 380\text{t},$$
$$W_{\text{fuel}} = 180.5\text{t}, \quad g = 9.81\text{m/s}^2, \quad w_p = 90\text{kg}, \quad n_p = 270$$

W_{fuel} was obtained from the listed maximum fuel capacity of 222.85m^3 and the mass density of jet fuel which is $\rho_{\text{fuel}} = 810\text{kg.m}^{-3}$. Replacing these in the Range Equation we obtain:

$$\boxed{R_{\max} = 16414\,\text{Kilometers}}$$

which is pretty close to the listed range of 16670 Kilometers.

(b) Imagine that the aerodynamic design can be improved such that L/D increases by 3%. How many more passengers could the airplane carry just from the perspective of maintaining the same maximum range? (Hint: Assume that the weight of each passenger with luggage is on average 90 kg).

Solution: The max range for the original design is:

$$R_{\max}^{(1)} = \frac{\left(\frac{L}{D}\right)^{(1)} h_f \eta_0 \ln\left(\frac{W_{\text{fuel}} + n_p w_p + W_s}{n_p w_p + W_s}\right)}{g}$$

And for the new design:

$$R_{\max}^{(2)} = \frac{\left(\frac{L}{D}\right)^{(2)} h_f \eta_0 \ln\left(\frac{W_{\text{fuel}} + w_p(\Delta n + n_p) + W_s}{w_p(\Delta n + n_p) + W_s}\right)}{g}$$

Since we are not extending the range, they should be the same and their ratio equal to 1:

$$\frac{R_{\max}^{(2)}}{R_{\max}^{(1)}} = 1 = 1.03 \frac{\ln\left(\frac{W_{\text{fuel}} + w_p(\Delta n + n_p) + W_s}{w_p(\Delta n + n_p) + W_s}\right)}{\ln\left(\frac{W_{\text{fuel}} + n_p w_p + W_s}{n_p w_p + W_s}\right)}$$

In this equation the only unknown is Δn, since all the parameters have been estimated above, except for W_s, which can be estimated from $MTOW, W_p = nw_p, W_{\text{fuel}}$ by:

$$W_s = MTOW - n_p w_p - W_{\text{fuel}} = 175.2\,\text{t}$$

However, we don't really need W_s or n, as they disappear once replaced in the equation above:

$$1 = 1.03\frac{\ln\left(\frac{\text{MTOW } + \ \Delta n \ w_p}{\text{MTOW } - \ W_{\text{fuel}} + \ \Delta n \ w_p}\right)}{\ln\left(\frac{\text{MTOW}}{\text{MTOW } - \ W_{\text{fuel}}}\right)}$$

Replacing the values above and solving for Δn, we obtain:

$$1 = 1.59839 \ln\left(\frac{\Delta\text{n}(90\text{kg}) + 380000.\text{kg}}{\Delta\text{n}(90\text{kg}) + 199500.\text{kg}}\right)$$

$$\boxed{\Delta n \sim 90}$$

(c) We would like to perform a sensitivity analysis of several parameters on the maximum range of the airplane. What would be the relative variation in the maximum range, *ceteris paribus*, for the following scenarios?

- A new design using an increased amount of composites in some of the main structural components would satisfy all the structural requirements while reducing the structure weight by 5%
- An architect reorganizes the space inside the aircraft, which can now carry 12 more passengers ($\simeq 5$ % more), without adding weight to the structure
- A new generation of engines allows to increase the propulsive efficiency by 5 %

Comment on your results.

Solution: In the first scenario:

$$W_s^{(3.1)} = 0.95 W_s^{(1)} = 168142\text{kg}$$

Replacing in the range equation and computing the ratio of the ranges of the new and the old design, we obtain:

$$\frac{R_{\max}^{(3.1)}}{R_{\max}^{(1)}} = \frac{\ln\left(\frac{W_{\text{fuel}} + n_p w_p + 0.95 W_s}{n_p w_p + 0.95 W_s}\right)}{\ln\left(\frac{W_{\text{fuel}} + n_p w_p + W_s}{n_p w_p + W_s}\right)} = 1.03385$$

i.e. approximately a 3.4 % increase in range.

In the second scenario:

$$\frac{R_{\max}^{(3.2)}}{R_{\max}^{(1)}} = \frac{\ln\left(\frac{W_{\text{fuel}} + (n_p + 12) w_p + W_s}{(n_p + 12) w_p + W_s}\right)}{\ln\left(\frac{W_{\text{fuel}} + n_p w_p + W_s}{n_p w_p + W_s}\right)} = 0.996$$

i.e. approximately a 0.4 % decrease in range.

The third scenario is trivial since the maximum range depends linearly on the propulsive efficiency. Therefore, increasing the propulsive efficiency by 5 % results in increasing the maximum range by 5 %. The range is therefore more sensitive to the propulsive efficiency than to the structure weight, and almost not sensitive to the number of passengers.

Problem M-1.2
(M.O.: M4)

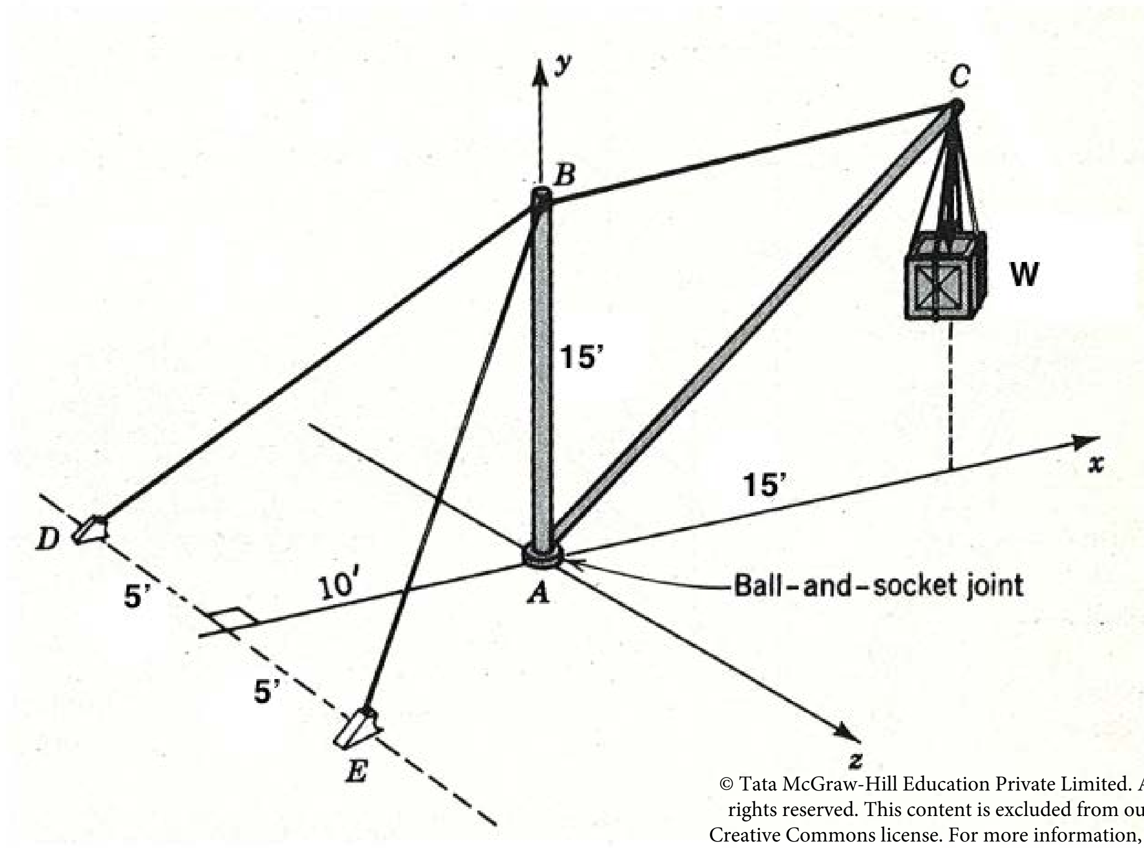


Figure 1: Crane Supporting a Weight

The crane shown in Figure 1 lifts a weight of W = 15,000 lb and is supported by cables BD and BE.

(a) Determine the cable tensions in BC, BD and BE (denote these as F_{BC}, F_{BD}, and F_{BE}, respectively).

Solution: We may obtain the cable tensions by considering the forces acting at certain points in the structure and enforcing equilibrium. A free body diagram of the forces at C is shown in Figure 2. Enforcing equilibrium in the x and y (or $\mathbf{e_1}$ and $\mathbf{e_2}$) directions

$$\sum F_y = 0 - W - \frac{\sqrt{2}}{2} F_{AC} = 0 \rightarrow F_{AC} = -\sqrt{2}W \tag{1}$$

$$\sum F_x = 0 - F_{BC} - \frac{\sqrt{2}}{2}F_{AC} = 0 \rightarrow F_{BC} = -\frac{\sqrt{2}}{2}F_{AC} = -\left(\frac{\sqrt{2}}{2}\right)(-\sqrt{2}W) = W \tag{2}$$

Thus, $\boxed{F_{BC} = W = 15,000\ lb}$.

Next, we apply equilbrium at point B. Due to symmetry (or balance of forces in the z-direction) forces $F_{BE} = F_{BD}$. Projecting the forces F_{BE} and F_{BD} in the x-direction and applying balance of forces in this direction gives

$$\sum F_x = 0 \tag{3}$$
$$-F_{BE}\cos\phi\cos\theta - F_{BD}\cos\phi\cos\theta + W = 0 \tag{4}$$
$$F_{BE}\cos\phi\cos\theta + F_{BD}\cos\phi\cos\theta = W \tag{5}$$
$$2F_{BE}\cos\phi\cos\theta = W \tag{6}$$

where we define the angles ϕ and θ in the planar triangles given in Figures 3 and 4. From the Figures, $\cos\phi = \frac{\sqrt{125}}{\sqrt{350}}$ and $\cos\theta = \frac{10}{\sqrt{125}}$. This gives

$$2F_{BE}\cos\phi\cos\theta = W \tag{7}$$
$$2F_{BE}\left(\frac{\sqrt{125}}{\sqrt{350}}\right)\left(\frac{10}{\sqrt{125}}\right) = W \tag{8}$$
$$F_{BE} = \frac{\sqrt{350}}{20}W = 14031 \text{ lb} \tag{9}$$

Thus, $\boxed{F_{BE} = F_{BD} = 14031 \text{ lb}}$.

(b) Assume that the crane may pivot about the xy-plane, so that cable BC now makes an angle α with the positive x-axis. Determine an expression for the ratio of cable tensions $\frac{F_{BD}}{F_{BE}}$ in terms of this angle α. What is the value of α when $\frac{F_{BD}}{F_{BE}} = 3$? *Note:* Follow the right-hand rule in selecting the positive convention for α.

Solution: A topdown view of cables BC, BE and BD is given in Figure 5. We apply summation of forces in both the x and z directions. This results in

$$\sum F_z = 0 \tag{10}$$
$$F_{BE}\cos\phi\sin\theta - F_{BD}\cos\phi\sin\theta - W\sin\alpha = 0 \tag{11}$$

$$\sum F_x = 0 \tag{12}$$
$$-F_{BE}\cos\phi\cos\theta - F_{BD}\cos\phi\cos\theta + W\cos\alpha = 0 \tag{13}$$

Thus, we aim to solve the following system of equations

$$F_{BE}\cos\phi\sin\theta - F_{BD}\cos\phi\sin\theta - W\sin\alpha = 0 \tag{14}$$
$$-F_{BE}\cos\phi\cos\theta - F_{BD}\cos\phi\cos\theta + W\cos\alpha = 0 \tag{15}$$

for the ratio $\frac{F_{BD}}{F_{BE}}$. We can use manipulations to obtain this ratio. One approach is to multiply the first equation by $\cos\theta$ and the second equation by $\sin\theta$. Adding the two equations (thereby eliminating F_{BE}) and solving for F_{BD} gives

$$F_{BD} = \frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{2\cos\phi\sin\theta\cos\theta}W \tag{16}$$

Subtracting the two equations (thereby eliminating F_{BD}) and solving for F_{BE} gives

$$F_{BE} = \frac{\sin\alpha\cos\theta + \cos\alpha\sin\theta}{2\cos\phi\sin\theta\cos\theta}W \tag{17}$$

Taking the ratios gives

$$\frac{F_{BD}}{F_{BE}} = \frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\sin\alpha\cos\theta + \cos\alpha\sin\theta} \tag{18}$$

Given that $\cos\theta = 10/\sqrt{125} = 2/\sqrt{5}$, $\sin\theta = 5/\sqrt{125} = 1/\sqrt{5}$ (see Figure 4), we have

$$\boxed{\frac{F_{BD}}{F_{BE}} = \frac{\cos\alpha - 2\sin\alpha}{2\sin\alpha + \cos\alpha}} \tag{19}$$

We may set $\frac{F_{BD}}{F_{BE}} = 3$ and solve for the corresponding value of α. This gives

$$3 = \frac{\cos\alpha - 2\sin\alpha}{2\sin\alpha + \cos\alpha} \tag{20}$$
$$6\sin\alpha + 3\cos\alpha = \cos\alpha - 2\sin\alpha \tag{21}$$
$$8\sin\alpha = -2\cos\alpha \tag{22}$$
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-2}{8} = \frac{-1}{4} \tag{23}$$

Thus, $\boxed{\alpha = \tan^{-1}(-1/4) \approx -14.0^\circ}$.

(c) For what value α will cable BE experience no loading? For what value α will cable BD experience no loading? Lastly, what will happen if α falls out outside this range?

Solution:

- If cable BE experiences no load, then cable BC is aligned in the same plane as triangle ABD so that cable BD experiences all of the load. This occurs when $\boxed{\alpha = -\tan^{-1}(5/10) = -26.57^\circ}$.
- If cable BD experiences no load, then $\boxed{\alpha = -\tan^{-1}(5/10) = 26.57^\circ}$ so that cable BC aligns with the plane of triangle ABE.
- Outside of this range of angles, one of the wires will feel a compressive force and start to buckle, so that the other wire must support the entire structure. If $\alpha < -26.57^\circ$, cable BE will be in compression. If $\alpha > 26.57^\circ$, cable BD will be in compression. Collapse may occur if one wire begins to buckle.

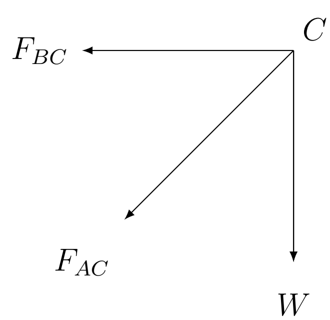


Figure 2: Free Body Diagram at Point C of Crane Structure

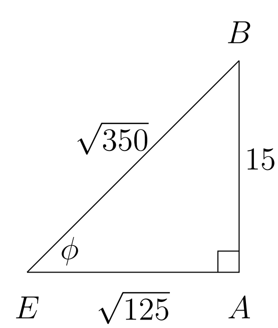


Figure 3: Triangle used for projection to xy-plane

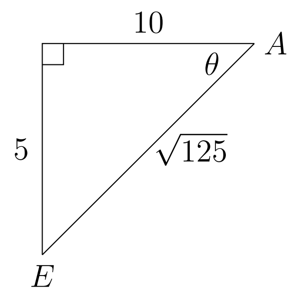


Figure 4: Triangle used for projection to x-axis

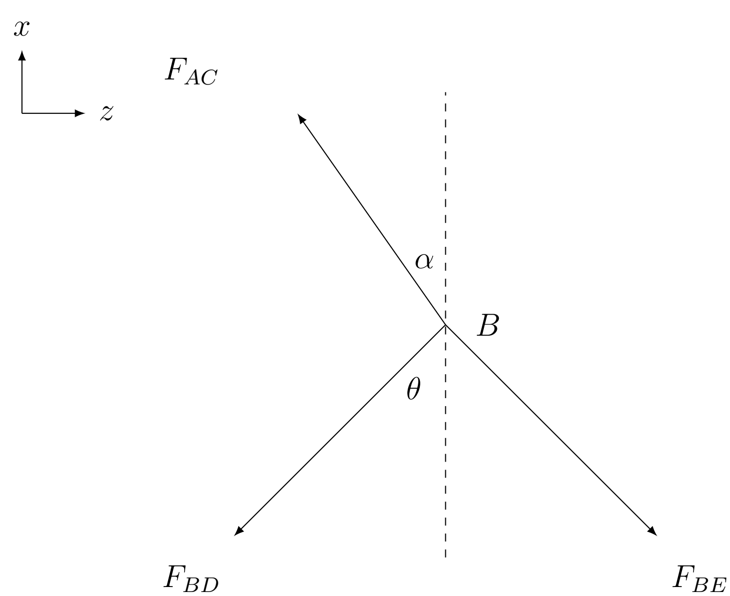


Figure 5: Top Down View of FBD at point B (part b)

Problem M-1.3

The top of a tin can is removed, and the empty can is inverted over a pair of billiard balls on a table as shown in the sketch.

Diameter of ball	45 mm
Weight of ball	2.0 N
Diameter of can	70 mm
Weight of empty can with lid removed	1.0 N

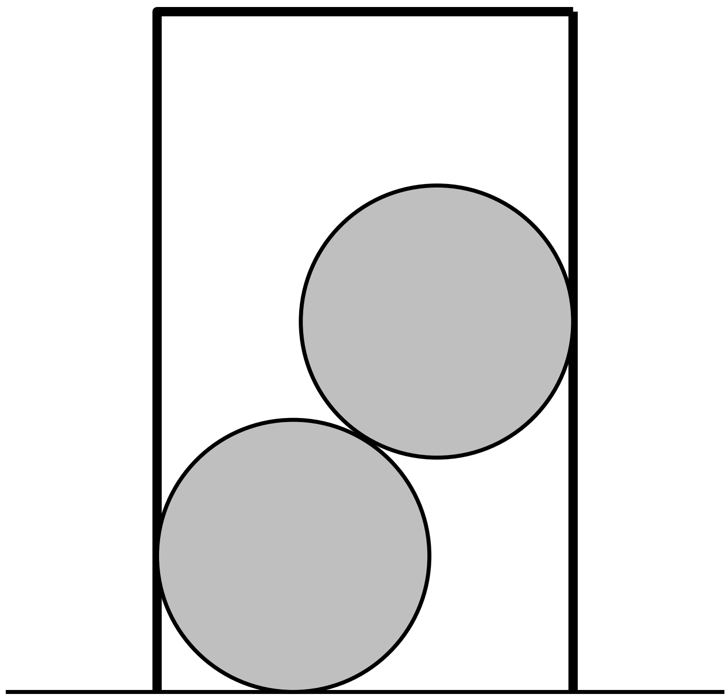
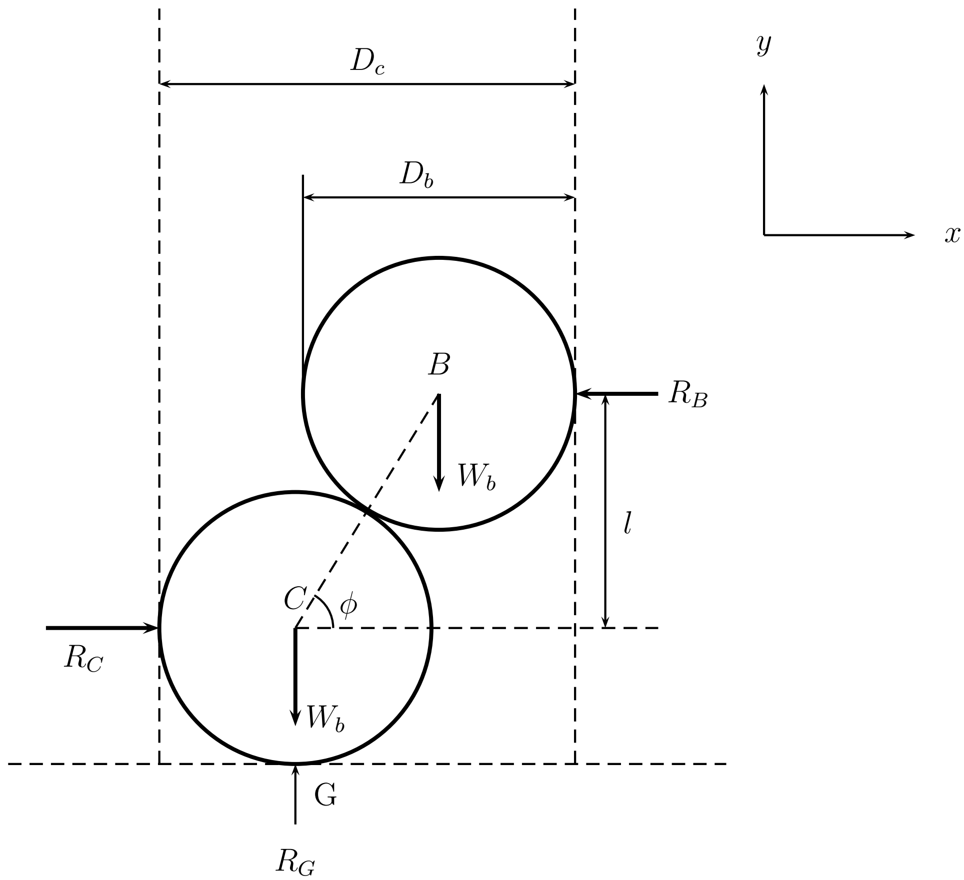


Figure 6: Billiard balls in a tin

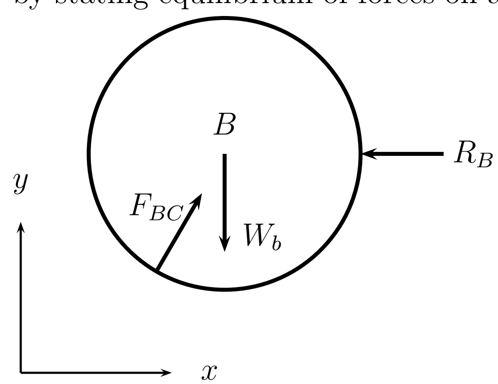
(a) For the parameters listed, is the system stable or will it tip over?

Solution:

In this problem we will place a ball B of weight W_b and diameter D_b on the top of ball C also of weight W_b and also with diameter D_b. The can will have a weight W_c and diameter D_c. The problem is illustrated by the following figure:



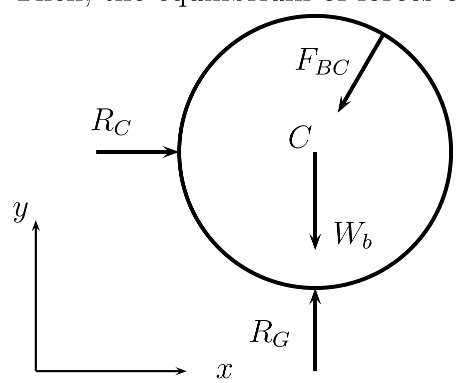
In this problem we recommend computing the forces and moments that would be required to attain static equilibrium. If those forces are not possible, then the system is not in equilibrium and is therefore unstable. We begin the problem by stating equilibrium of forces on all balls in the system. Starting with ball B:



$$\sum_{F_x} : F_{BC} * \cos\phi - R_B = 0 \tag{24}$$

$$\sum_{F_y} : F_{BC} * \sin\phi - W_b = 0 \tag{25}$$

Then, the equilibrium of forces on ball C gives:



$$\sum_{F_x} : R_C - F_{BC} * \cos\phi = 0 \tag{26}$$

$$\sum_{F_y} : R_G - F_{BC} * \sin\phi - W_b = 0 \tag{27}$$

Note that we used Newton's third law to equal the forces of B on C and of C on B. We have four equations and four unknowns R_B, R_C, R_G, and F_{BC}, and we can therefore solve the problem. The values of $cos(\phi)$ and $sin(\phi)$ can be evaluated with the geometry:

$$\cos\phi = \frac{D_c - D_b}{D_b} \tag{28}$$

$$\sin\phi = \sqrt{1 - \cos\phi^2} \tag{29}$$

The algebraic computation produces the following results which for convenience are stated in terms of the variable (ϕ):

$$R_G = 2W_b \tag{30}$$

$$R_B = \frac{W_b}{\tan\phi} \tag{31}$$

$$R_C = \frac{W_b}{\tan\phi} \tag{32}$$

$$F_{CB} = F_{BC} = \frac{W_b}{\sin\phi} \tag{33}$$

Now we can look at the equilibrium of forces on the can. Along the horizontal axis, we have:

$$\sum_{F_x} : R_B - R_C = 0 \tag{34}$$

The equilibrium of forces in the horizontal direction is satisfied based on the values of R_B and R_C obtained previously. The equilibrium in the y direction is computed assuming that the normal force acting on the rim of the can in the x-z plane will effectively act along the axis of symmetry. We therefore describe this ring of normal forces per unit length as being represented by an effective force on the left most point and an effective force on the right most point. We will use force and moment equilibrium to compute the magnitude of these two fictitious forces that would be required to hold this configuration in static equilibrium.

$$\sum_{F_y} = F_{LeftRim} + F_{RightRim} - W_c = 0 \tag{35}$$

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + R_C * \frac{D_b}{2} - R_B * (\frac{D_b}{2} + D_b * \sin\phi) = 0 \tag{36}$$

Here we have taken the moments about the origin and we can substitute our previous expressions to compute the moments on the can as a function of problem parameters. The previous expression simplifies as:

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} - W_b * D_b * \cos\phi = 0 \tag{37}$$

We use values given for the problem statement, together with the value $cos(\phi) = 0.555$ computed, and we obtain:

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * 0.035 - 0.05 = 0 \tag{38}$$

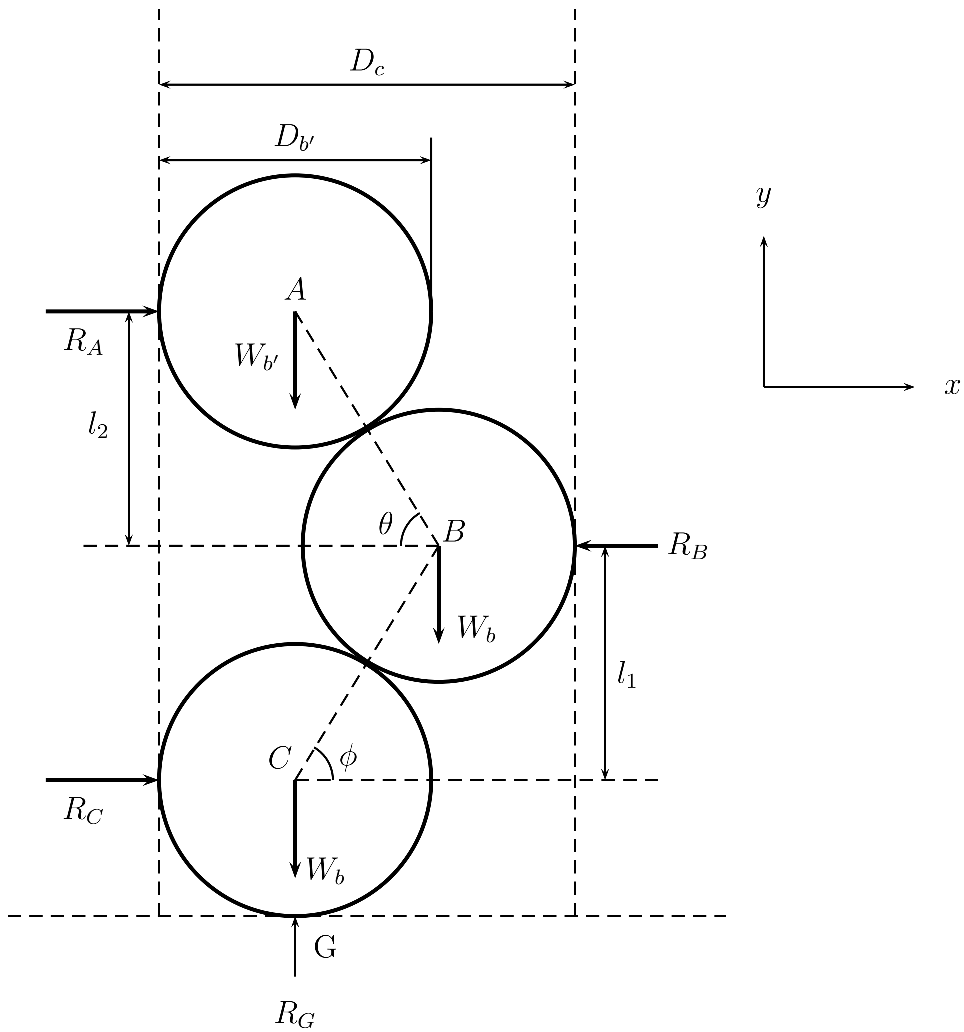
$$\sum_{F_y} = F_{LeftRim} + F_{RightRim} - 1 = 0 \tag{39}$$

We obtain a system with two equations and two unknowns, that we can solve. The results give $F_{RightRim} = 1.214$ N and $F_{LeftRim} = -0.214$ N in order to hold this system in static equilibrium. Of course it is not possible for the contact on the left side of the can to produce a vertical force of negative value. Normal forces always act normal to the surface so a force acting in the negative y direction from a surface with a normal in the positive y direction is not possible. The can would need to be glue or held down on the left hand side to attain static equilibrium therefore the configuration as described is not stable.

(b) Will a third ball added on top of these two provide a stabilizing force?

Solution:

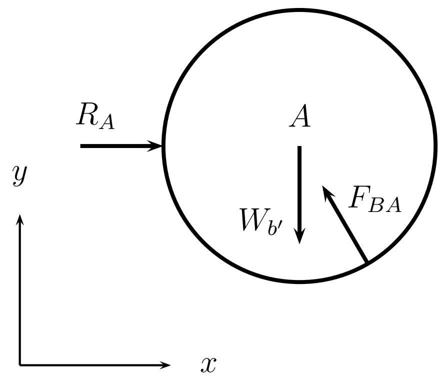
In this section we analyze the effect that the addition of a third ball has on the system. The system can now be represented the following way:



We start by stating the equilibrium of forces on all balls in the system. Starting with ball A this time we have:

$$\sum_{F_x} : R_A - F_{BA} * \cos\theta = 0 \tag{40}$$

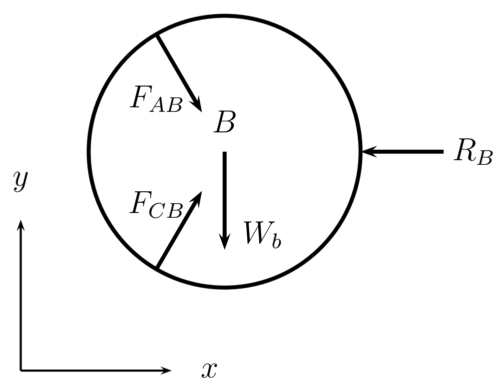
$$\sum_{F_y} : F_{BA} * \sin\theta - W_{b'} = 0 \tag{41}$$



Then for ball B:

$$\sum_{F_x} : F_{AB} * \cos\theta + F_{CB} * \cos\phi - R_B = 0 \tag{42}$$

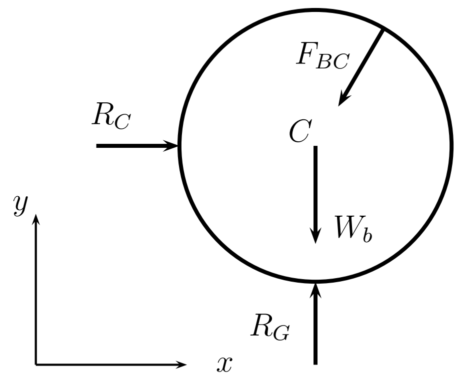
$$\sum_{F_y} : F_{CB} * \sin\phi - F_{AB} * \sin\theta - W_b = 0 \tag{43}$$



And finally for ball C:

$$\sum_{F_x} : R_C - F_{BC} * \cos\phi = 0 \tag{44}$$

$$\sum_{F_y} : R_G - F_{BC} * \sin\phi - W_b = 0 \tag{45}$$



We also utilize newtons third law to state that the magnitude of the corresponding contact forces between the balls are equal. We have six equations and six unknowns, which are R_A, R_B, R_C, R_G, F_{AB}, and F_{BC}, so we can solve the system. In order to do so we need to describe the relevant geometric quantities in terms of problem parameters.

$$\cos\phi = \frac{D_c - D_b}{D_b} \tag{46}$$

$$\cos\theta = \frac{D_c - \frac{D_{b'}+D_b}{2}}{\frac{D_{b'}+D_b}{2}} \tag{47}$$

$$\sin\theta = \sqrt{1 - \cos\theta^2} \tag{48}$$

$$\sin\phi = \sqrt{1 - \cos\phi^2} \tag{49}$$

The algebraic computation produces the following results which for convenience are stated in terms of the variables θ and ϕ.

$$R_G = 2 * W_b + W_{b'} \tag{50}$$

$$R_A = \frac{W_{b'}}{\tan\theta} \tag{51}$$

$$R_B = \frac{W_{b'}}{\tan\theta} + \frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi} \tag{52}$$

$$R_C = \frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi} \tag{53}$$

$$F_{AB} = F_{BA} = \frac{W_{b'}}{\sin\theta} \tag{54}$$

$$F_{BC} = F_{CB} = \frac{W_{b'} + W_b}{\sin\phi} \tag{55}$$

With this information we can determine the forces acting on the can

$$\sum_{F_x} : R_B - R_C - R_A = 0 \tag{56}$$

This condition is satisfied by substitution of the previously computed reaction force magnitudes.

$$\sum_{F_x} : (\frac{W_{b'}}{\tan\theta} + \frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi}) - (\frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi}) - \frac{W_{b'}}{\tan\theta} = 0 \tag{57}$$

The equilibrium in the y direction is computed assuming that the normal force acting on the rim of the can in the x-z plane will effectively act along the axis of symmetry. We therefore describe this ring of normal forces per unit length as being represented by an effective force on the left most point and an effective force on the right most point. We will use force and moment equilibrium to compute the magnitude of these two fictitious forces that would be required to hold this configuration in static equilibrium.

$$\sum_{F_y} = F_{LeftRim} + F_{RightRim} - W_c = 0 \tag{58}$$

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + R_A * (\frac{D_b}{2} + D_b * \sin\phi + \frac{D_b + D_{b'}}{2} * \sin\theta) \tag{59}$$

$$+R_C * \frac{D_b}{2} - R_B * (\frac{D_b}{2} + D_b * \sin\phi) = 0 \tag{60}$$

Here we have taken the moments about the origin and we can substitute our previous expressions to compute the left and right normal forces on the can as a function of problem parameters.

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (\frac{W_{b'}}{\tan\theta}) * (\frac{D_b}{2} + D_b * \sin\phi + \frac{D_b + D_{b'}}{2} * \sin\theta) \tag{61}$$

$$+(\frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi}) * \frac{D_b}{2} - (\frac{W_{b'}}{\tan\theta} + \frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi}) * (\frac{D_b}{2} + D_b * \sin\phi) = 0 \tag{62}$$

Then we cancel terms

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (\frac{W_{b'}}{\tan\theta}) * (\frac{D_b + D_{b'}}{2} * \sin\theta) \tag{63}$$

$$-(\frac{W_{b'}}{\tan\phi} + \frac{W_b}{\tan\phi}) * D_b * \sin\phi = 0 \tag{64}$$

Simplify

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (W_{b'}) * (\frac{D_b + D_{b'}}{2} * \cos\theta) \quad (65)$$

$$-(W_{b'} + W_b) * D_b * \cos\phi = 0 \quad (66)$$

Observe that the third ball has a radius equivalent to that of the other two so therefore $D_b = D_{b'}$ and also $\theta = \phi$. Substituting this into our moment expression we have the following.

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (W_{b'}) * (\frac{D_b + D_{b'}}{2} * \cos\theta) \quad (67)$$

$$-(W_{b'} + W_b) * D_b * \cos\phi = 0 \quad (68)$$

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (W_{b'}) * (D_b * \cos\phi) - (W_{b'} + W_b) * D_b * \cos\phi = 0 \quad (69)$$

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} - (W_b) * D_b * \cos\phi = 0 \quad (70)$$

Observe that this expression is the same as computed in part A. Our analysis will be the same and indicate that the third ball does not stabilize the can. In fact, this expression is entirely independent of $W_{b'}$ therefore it does not matter how heavy our third ball is! No ball of diameter 45 mm can stabilize this system no matter its weight!

(c) Can any changes to this third ball's weight or radius be made to stabilize this system?

Solution: To understand the impact of balls of various sizes and weights we will simplify the general expression for the moments on the can by substituting in the definitions of $\cos\theta$ and $\cos\phi$.

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (W_{b'}) * (\frac{D_b + D_{b'}}{2} * (\frac{D_c - \frac{D_{b'}+D_b}{2}}{\frac{D_{b'}+D_b}{2}}))- \tag{71}$$

$$(W_{b'} + W_b) * D_b * (\frac{D_c - D_b}{D_b}) = 0 \tag{72}$$

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (W_{b'}) * (D_c - \frac{D_{b'} + D_b}{2}) \tag{73}$$

$$-(W_{b'} + W_b) * (D_c - D_b) = 0 \tag{74}$$

$$\sum_{M_z} = (F_{RightRim} - F_{LeftRim}) * \frac{D_c}{2} + (W_{b'}) * (\frac{D_b - D_{b'}}{2}) - (W_b) * (D_c - D_b) = 0 \tag{75}$$

The middle term is the term containing the parameters of the third ball. It vanishes when the diameters are equal, but can provide a positive moment to the system if $D_{b'} < D_b$. This moment is proportional to the weight of the new ball due to dimensional scaling laws. From part A we find that an additional .015 Nm of torque was required to hold the system in equilibrium. Therefore the constraint that the third ball must meet is the following.

$$W_{b'} * (\frac{D_b - D_{b'}}{2}) > 0.015 \tag{76}$$

$$W_{b'} * (\frac{0.045 - D_{b'}}{2}) > 0.015 \tag{77}$$

Any suggested ball that meets this constraint would work to stabilize the system, but there is an additional condition that $D_{b'} > 0.025$ in order for the third ball to avoid sliding past ball B and make contact with ball C. Should this occur our analysis of the contact problem would be incorrect as the contact points and geometry would be different.

Additionally, it should be noted that with this smaller ball diameter it is possible for the small third ball to become lodged on the right hand side of the can. Such a situation would not help to stabilize the can as the new ball would effectively be increasing the weight of ball B and consequently the force at reaction B. The new ball would also be providing an addition to the moment arm for the horizontal forces on the right side of the can which convey destabilizing, negative moments. It is interesting to note that the larger ball will always destabilize, and a smaller ball can provide either effect depending on how it settles. Lastly, we should also note that if the mass of this new third ball becomes sufficiently large and the new diameter is sufficiently smaller then the diameter of the balls below it, the new positive moment term could potentially rotate the can around the left rim and cause instability in the other direction.

○ **Problem M-1.4**

A small box of mass m hangs from three massless ropes as shown in Figure 7. The ropes are attached to a ball joint which is located at the origin O of the Cartesian coordinate system Oxyz. Two of these ropes are also attached to the ball joints A and B which are fixed to a rigid wall; the third one is connected to the ball joint C which is attached to a rigid block of mass M. The latter stands on a plane surface; the coefficient of static friction between it and the surface is μ_s. Note that the gravity g acts on the small box and the block which are both made from a homogeneous material.

Hint:

- Static friction is the force F that acts between two solid surfaces which are in contact but do not move relative to each other. It opposes the relative lateral motion of the two surfaces which would occur in the absence of the friction due to applied external forces. The static friction force acts tangentially to the two contacting surfaces. According to the Coulomb model, it fulfills the condition

$$|F| \leq \mu_s N$$

  where μ_s is the coefficient of static friction corresponding to the considered pair of surfaces, and N is the acting normal force between the two surfaces. (If the condition was violated, the surfaces would start moving relative to each other.)

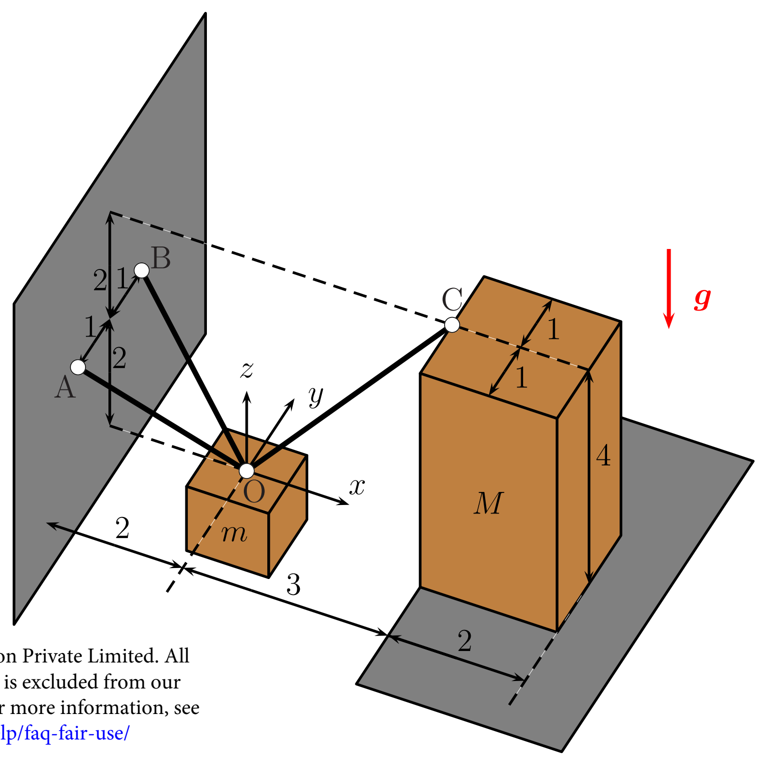


Figure 7: Box hanging from three ropes.

Questions:

(a) For safety reasons, the magnitude of the tensile force in the ropes must not exceed the maximum admissible value $F_{\max}$. What is the maximum value for the mass m of the small box so that this safety condition is met?

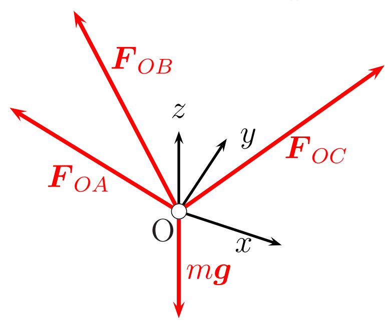
Solution: Ropes can support only **axial** tensile forces. Express these forces as vectors in terms of the sought force magnitudes and **unit** direction vectors:

$$\text{Force in Rope OA:} \quad \boldsymbol{F}_{OA} = F_{OA}\,\hat{\mathbf{v}}_{OA} \quad \text{where} \quad \hat{\mathbf{v}}_{OA} = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \tag{78}$$

$$\text{Force in Rope OB:} \quad \boldsymbol{F}_{OB} = F_{OB}\,\hat{\mathbf{v}}_{OB} \quad \text{where} \quad \hat{\mathbf{v}}_{OB} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \tag{79}$$

$$\text{Force in Rope OC:} \quad \boldsymbol{F}_{OC} = F_{OC}\,\hat{\mathbf{v}}_{OC} \quad \text{where} \quad \hat{\mathbf{v}}_{OC} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \tag{80}$$

Consider the force equilibrium for the ball joint at Point O:



$$\sum \boldsymbol{F} = \mathbf{0}:$$

$$\boldsymbol{F}_{OA} + \boldsymbol{F}_{OB} + \boldsymbol{F}_{OC} + m\boldsymbol{g} = \mathbf{0} \tag{81}$$

$$\text{where} \quad \boldsymbol{g} = g\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{82}$$

Inserting Eqs. (78) to (80) into Eq. (81) and solving for F_{OA}, F_{OB}, F_{OC} yields:

$$F_{OA} = F_{OB} = \frac{9}{28}\,mg \tag{83}$$

$$F_{OC} = \frac{5}{7}\,mg \tag{84}$$

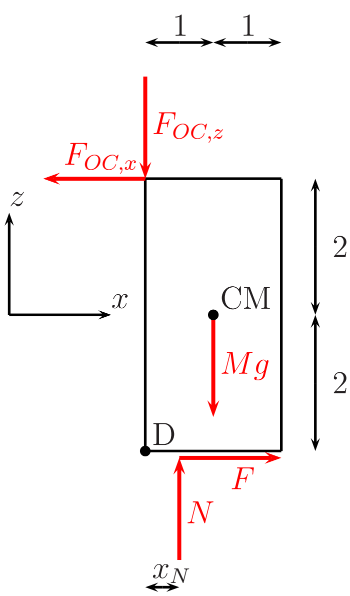
The largest force occurs in Rope OC which connects the small box to the block. Setting $F_{OC} = F_{\max}$ then yields the maximum value for the mass m that the small box can have while the force $F_{\max}$ is not exceeded in any rope:

$$F_{OC} = F_{\max} \quad \Rightarrow \quad m_{\max} = \frac{7}{5}\,\frac{F_{\max}}{g} \tag{85}$$

(b) If the mass m of the small box was chosen too large, the block of mass M might tilt over. Assuming that the coefficient of static friction between the block and the surface is large enough to prevent sliding, what is the maximum value for the mass m of the box at which no tilting of the block occurs?

Solution: In order to approach this part, consider the static equilibrium for the block using the x- and z-components of the force vector $\boldsymbol{F}_{OC}$ (i.e. $F_{OC,x} = \frac{3}{7}\,mg$ and $F_{OC,z} = \frac{4}{7}\,mg$) from the previous part.

Both a resultant normal force N (resulting from distributed normal forces exerted from the ground on the block) and a resultant friction force F must be included to represent the ground. Note that the point of attack of these forces is not necessarily below the center of mass (CM) due to the force exerted by Rope OC but at an a priori unknown position x_N:



$$\sum F_x = 0: \quad -F_{OC,x} + F = 0 \tag{86}$$

$$\sum F_z = 0: \quad -F_{OC,z} - Mg + N = 0 \tag{87}$$

$$\sum M^{(D)} = 0: \quad \text{(counterclockwisely)}$$

$$F_{OC,x} \cdot 4 - Mg \cdot 1 + N \cdot x_N = 0 \tag{88}$$

The equilibrium conditions yield the unknown reaction forces N and F as well as the unknown distance x_N:

$$F = \frac{3}{7}mg \tag{89}$$

$$N = Mg + \frac{4}{7}mg \tag{90}$$

$$x_N = \frac{1 - \frac{12}{7}\frac{m}{M}}{1 + \frac{4}{7}\frac{m}{M}} \tag{91}$$

The block does not tilt if $x_N > 0$. In that case, the moment about D resulting from both the force in Rope OC **and** the weight is positive in the clockwise direction. If $x_N = 0$, this moment would become zero, and the block would start tilting as soon as F_{OC} was increased slightly more. Therefore, the maximum mass m of the small box for which no tilting of the block occurs is

$$x_N = 0 \quad \Rightarrow \quad m = \frac{7}{12}M. \tag{92}$$

(c) For which value of the coefficient of static friction μ_s would the block start sliding (assuming that no tilting has occurred yet)? Using the result from the previous part, how should the value of the coefficient of static friction μ_s be chosen so that sliding and tilting would occur for the same value of the mass m?

Solution:
Following the hint given in the problem statement, the condition for no slipping of the block is

$$|F| \leq \mu_s N, \tag{93}$$

meaning that slipping would start once

$$|F| = \mu_s N \tag{94}$$

has been reached. Using the values for F and N obtained in the previous part, Eq. (94) yields the minimum coefficient of static friction required to prevent slipping of the block,

$$\mu_s = \frac{3}{4 + 7\frac{M}{m}}. \tag{95}$$

If the onset of tilting and slipping of the block is supposed to occur for the same value of m, the result (92) from the previous part needs to be used in Eq. (95):

$$\mu_s = \frac{3}{16} \tag{96}$$

○ **Problem M-1.5**

The circular cylinder A rests on top of two half-circular cylinders B and C, all having the same radius r, as shown in Figure 8. The weight of A is W, and the weight of B and C is $\frac{1}{2}W$ each. All cylinders are made from a homogeneous material. Assume that the coefficient of friction between the flat surfaces of the half-cylinders and the horizontal table top is f. Determine the maximum distance d between the centers of the half-cylinders to maintain equilibrium.

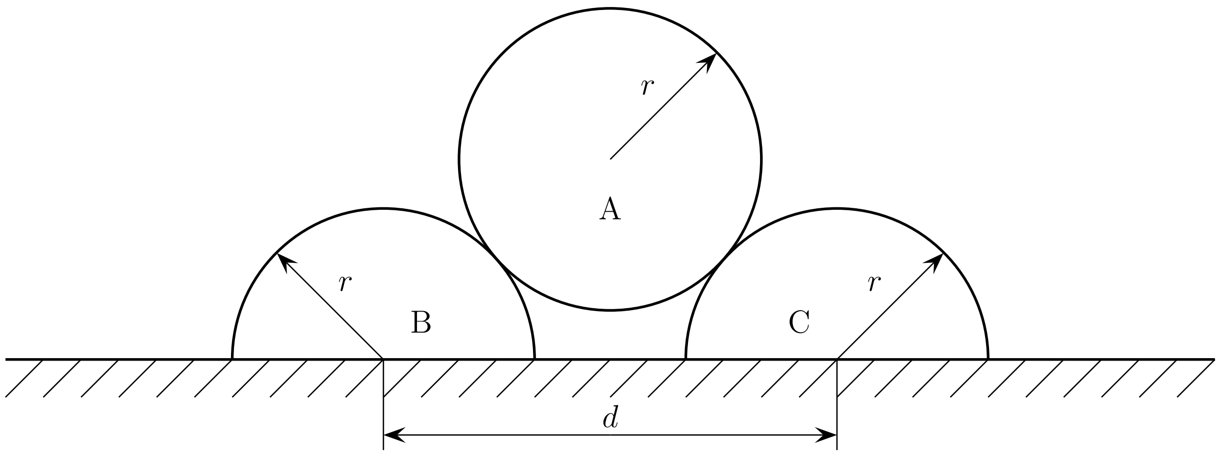
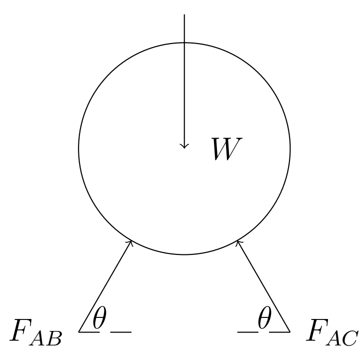


Figure 8: Circular cylinder A resting on half cylinders B and C.

Solution: We begin by drawing a free body diagram of cylinder A, shown in the figure



Applying force equilibrium in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions

- Equilibrium in $\mathbf{e}_1$-direction for cylinder A

$$\sum F_1 = 0 \tag{97}$$

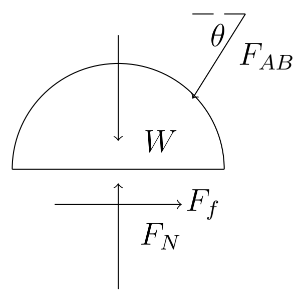
$$F_{AB}\cos\theta - F_{BC}\cos\theta = 0 \tag{98}$$

$$F_{AB} = F_{BC} \tag{99}$$

- Equilibrium in $\mathbf{e}_2$-direction for cylinder A

$$\sum F_2 = 0 \tag{100}$$
$$-W + F_{AB}\sin\theta + F_{BC}\sin\theta = 0 \tag{101}$$
$$-W + 2F_{AB}\sin\theta = 0 \tag{102}$$
$$F_{AB} = F_{BC} = \frac{W}{2\sin\theta} \tag{103}$$

We may now analyze the forces acting on either half-cylinder B or C. We choose to analyze B, a free body diagram of which is given in the following figure (including the normal and friction forces acting on it).



Applying force equilibrium in the $\mathbf{e}_2$ and $\mathbf{e}_1$ directions

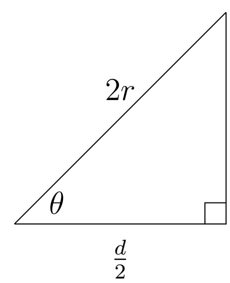
- Equilibrium in $\mathbf{e}_2$-direction for half-cylinder B

$$\sum F_2 = 0 \tag{104}$$
$$F_N - \frac{W}{2} - F_{AB}\sin\theta = 0 \tag{105}$$
$$F_N = \frac{W}{2} + F_{AB}\sin\theta \tag{106}$$
$$F_N = \frac{W}{2} + \frac{W}{2\sin\theta}\sin\theta \tag{107}$$
$$F_N = W \tag{108}$$

- Equilibrium in $\mathbf{e}_1$-direction for half-cylinder B

$$\sum F_1 = 0 \tag{109}$$
$$F_{AB}\cos\theta - F_f = 0 \tag{110}$$
$$\frac{W}{2\sin\theta}\cos\theta - fF_N = 0 \tag{111}$$
$$\frac{W}{2}\cot\theta - fW = 0 \tag{112}$$
$$\cot\theta = 2f \tag{113}$$

We may determine $\cot\theta$ in terms of geometric quantities by analyzing the triangle in the following figure.



Using the Pythagorean theorem, we find that the missing side has length $\sqrt{4r^2 - \frac{d^2}{4}}$. Thus,

$$\cot\theta = 2f \tag{114}$$

$$\frac{d/2}{\sqrt{4r^2 - \frac{d^2}{4}}} = 2f \tag{115}$$

$$\frac{d^2/4}{4r^2 - \frac{d^2}{4}} = 4f^2 \tag{116}$$

$$d^2 = 16f^2(4r^2 - \frac{d^2}{4}) \tag{117}$$

Gathering like terms and simplifying results in the following expression for the maximum distance d

$$\boxed{d = \frac{8rf}{\sqrt{1+4f^2}}} \tag{118}$$