Buckling of Beams

Reading: CDL Sections 9.1 to 9.4
Optional: CDL Sections 9.5, 9.6

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Outline

1. Buckling of Beams
   - Learning objectives
   - Introduction to bifurcation of equilibrium and structural instability
   - A single degree-of-freedom example
   - Columns
   - Column buckling (Euler buckling)
   - Example: Simply-supported column
   - Other boundary conditions
   - Effects of initial imperfections
   - Failure of columns

- Understand the basic concept of structural instability and bifurcation of equilibrium.
- Derive the basic buckling load of beams subject to uniform compression and different displacement boundary conditions.
- Understand under what conditions structural design is limited by buckling considerations.
- Understand the response of beam structures under a combination of tranverse loads and intense compressive loads.
- Understand the postbuckling behavior of beam structures.

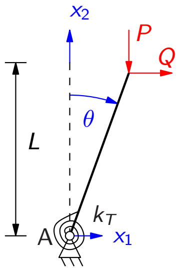
Our structural theories so far are all based on the linear theory of elasticity and have therefore furnished unique solutions. All along, one of the key assumptions was that equilibrium could be stated in the undeformed configuration of the structure and that the results would also hold in the deformed configuration. Structural instabilities have their origin in nonlinear effects that arise as a consequence of dropping this assumption and stating equilibrium in the deformed configuration.

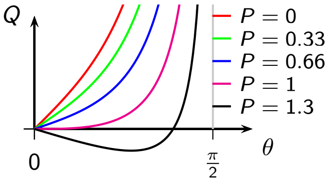
The simplest and perhaps most important instability from the standpoint of Aerospace engineering is the Buckling of Slender Beams or Columns under compressive loads.

Consider a rigid rod of length L hinged at Point A where a torsional spring of stiffness k_T provides the only deformable element of the system according to the relation $M(\theta) = k_T\theta$ between the internal torque on the spring and the rotation angle θ.
The beam is loaded with two concentrated loads Q and P, as shown in the figure.
We are interested in analyzing the equilibrium of the bar in the deformed configuration. In this case, this simply requires matching the external and internal moments with respect to A, which gives:

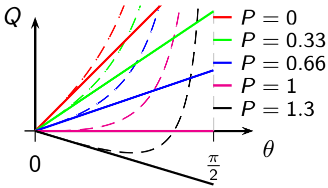
$$Q\cos(\theta)L \; + \; P\sin(\theta)L \; = k_T\theta$$

This gives the full nonlinear response. The figure on the right shows $Q = Q(\theta)$ for different values of P and $k_T = 1, L = 1$.





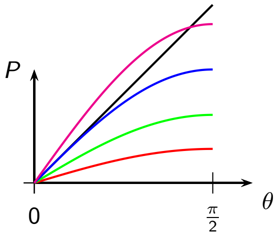
We are not interested in the nonlinear behavior for large θ, where Q grows unbounded because it loses it's moment arm, but rather, in the response for very small θ, which is almost linear. The linearized solution is obtained by simply approximating $\cos\theta \sim 1, \sin\theta \sim \theta$, which gives:



$$Q = \left(\frac{k_T}{L} - P\right)\theta$$

We can see that the main effect of the compressive load P is not to "drive" the deformation but to decrease the stiffness of the system, since it reduces the slope of the response curve at the origin. In the linearized model, when $P = P_{cr} = \frac{k_T}{L}$, the system has no stiffness and it can be in equilibrium for any value of θ under zero lateral load Q, which means that the structure is unstable. For $P > P_{cr}$, the effective stiffness of the structure is negative (what does this mean?). We therefore see that it is the compressive force P, that affects the stability of our structure, the lateral force Q does not play a role. A more useful view is then to consider the autonomous (undriven) system ($Q = 0$), and explore how P affects the characteristics of the system (stiffness).

$$P\sin(\theta) = \frac{k_T}{L}\theta$$



This is the essence of the analysis of
linearized stability of structures:

- state equilibrium in the deformed configuration
- linearize the equations
- study the autonomous (undriven) system loss of stiffness

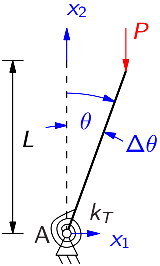
- This can be rearranged as

$$Q = \underbrace{\left(\frac{k_T - PL}{L} \right)}_{= k_{\text{eff}}} \theta = k_{\text{eff}}\, \theta$$

  where k_{eff} is an **effective torsional stiffness**.
- **Note**: Here, the load affects the stiffness; as P is being increased, k_{eff} decreases.
- **Important value** for P: If $P = k_T/L$, then $k_{\text{eff}} = 0$!
  This is the point of "**static instability**" or "**buckling**".
- Terminology:
  eigenvalue = value of load for static instability
  eigenvector = displacement shape / mode of structure
  (These terms will be revisited later.)

- Now, assume that only P acts and the rod is in a position θ_{eq} of static equilibrium.
- Then, perturb θ_{eq} by a small amount $\Delta\theta$, and release the rod.
  - If the rod returns to θ_{eq}: The position θ_{eq} is a **stable equilibrium**.
  - If the rod moves away from θ_{eq}: The position θ_{eq} is an **unstable equilibrium**.

For instance, consider $\theta_{\text{eq}} = 0$ and $\Delta\theta > 0$ so that $\theta = \theta_{\text{eq}} + \Delta\theta = \Delta\theta$:



$$\sum M^{(A)} = 0 : \quad \Rightarrow \quad -P\sin(\Delta\theta)L + k_T\Delta\theta = -\underbrace{I_A}_{>0}\ddot{\theta}$$

$$\Rightarrow \quad -I_A\ddot{\theta} = (k_T - PL)\,\Delta\theta$$

Note:
$-I_A\ddot{\theta}$ acts **counterclockwisely** (towards $\theta_{\text{eq}} = 0$)
$+I_A\ddot{\theta}$ acts **clockwisely** (away from $\theta_{\text{eq}} = 0$)

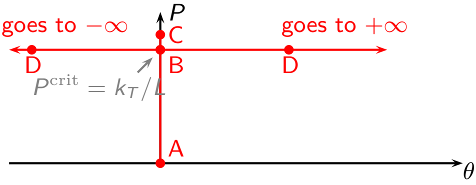
Therefore, if the rod is released from $\theta = \Delta\theta$:

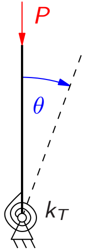
- If $k_T > PL \Rightarrow$ **stable**, $\theta \to \theta_{\text{eq}} = 0$ (due to friction)
- If $k_T < PL \Rightarrow$ **unstable**, $\theta \to \infty$
  Critical point:

$$\boxed{P^{\text{crit}} = \frac{k_T}{L}}$$

Spring cannot provide a sufficient restoring force.

Response of rod with torsional spring to compressive load P:

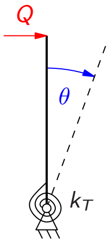


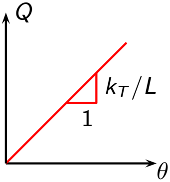


ABC	equilibrium path but not stable
ABD	equilibrium path, deflection grows unbounded ("bifurcation")
	(B is bifurcation point; 2 possible equilibrium paths)

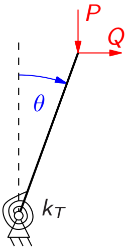
Note: If P is negative, the stiffness will increase.

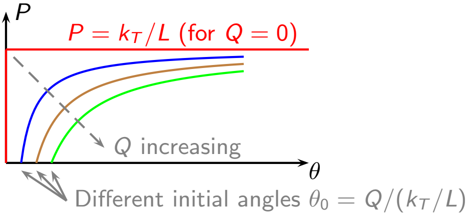
Response of rod with torsional spring to perpendicular load Q:





Response of rod with torsional spring to loads along and perpendicular to rod:

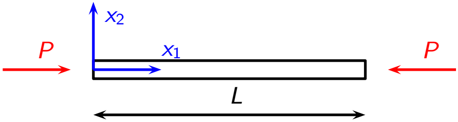




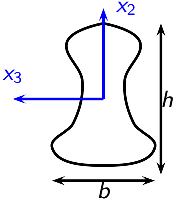
Note:

1. If Q and P are removed prior to instability, spring brings bar back to original configuration (as structural stiffnesses do for various configurations).
2. Bifurcation is a mathematical concept. The manifestations in actual systems are altered due to physical realities/imperfections.

Basic geometry of beam/column:



General symmetric cross-section:

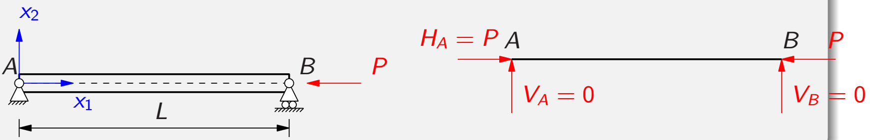


Assumptions:

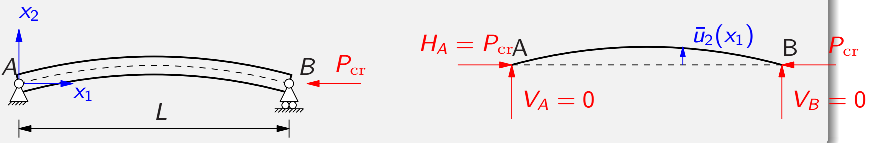
- slender: $L \gg b, h$
- constant cross-section (EI is constant)

A basic question we are trying to answer in the analysis of the buckling of beams is whether there exist deformed configurations under which the beam can stay deformed in equilibrium under the action of the compressive load P. If that is the case, what is the deformed configuration shape and what is the value of the load P_{cr}?

Undeflected column

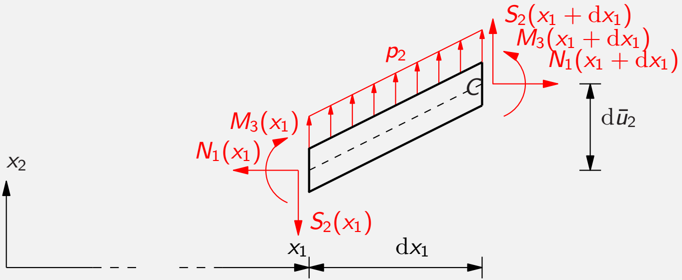


Deflected column



We saw that the key ingredient in the analysis of structural instability is to formulate equilibrium in the deformed configuration, where the axial compressive load contributes to the external bending moment, thus reducing the effective bending stiffness of the structure

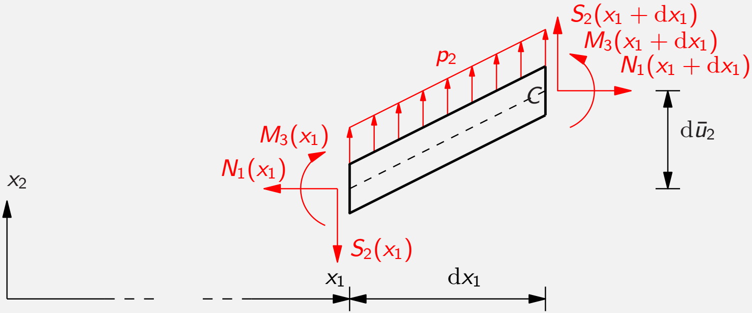
Equilibrium of a differential element in the deformed configuration



$$\sum F_{x_1} = 0 : \quad N_1(x_1 + \mathrm{d}x_1) - N_1(x_1) = 0 \Rightarrow \frac{\mathrm{d}N_1}{\mathrm{d}x_1} = 0$$

$$\sum F_{x_2} = 0 : \quad S_2(x_1 + \mathrm{d}x_1) - S_2(x_1) + p_2\,\mathrm{d}x_1 = 0 \Rightarrow \frac{\mathrm{d}S_2}{\mathrm{d}x_1} = -p_2$$

Equilibrium of a differential element in the deformed configuration



$$\sum M_3^{(C)} = 0 : \quad M_3(x_1 + \mathrm{d}x_1) - M_3(x_1) + \mathrm{d}x_1 S_2(x_1)$$

$$- \mathrm{d}\bar{u}_2 N_1(x_1) = 0$$

$$\Rightarrow \frac{\mathrm{d}M_3}{\mathrm{d}x_1} + S_2 - N_1 \frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1} = 0$$

Static equilibrium

$$\boxed{\frac{\mathrm{d}N_1}{\mathrm{d}x_1} = 0} \qquad \boxed{\frac{\mathrm{d}S_2}{\mathrm{d}x_1} = -p_2} \qquad \boxed{\frac{\mathrm{d}M_3}{\mathrm{d}x_1} + S_2 - N_1\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1} = 0} \tag{1}$$

Compatibility + Constitutive

$$\boxed{M_3 = EI\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}} \tag{2}$$

Fourth-order ODE for *EI* constant

$$\boxed{EI\frac{\mathrm{d}^4\bar{u}_2}{\mathrm{d}x_1^4} - N_1\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = p_2} \tag{3}$$

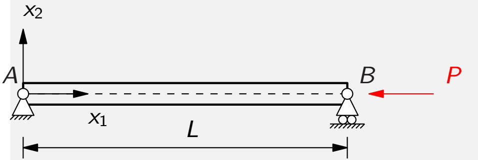
Note: The additional term (in red) arises when stating equilibrium in the deformed configuration under the assumption of "moderately large" deflections. Also, from (1) and (2), we note that the shear force also depends on he deformation in a different way from beam theory:

Shear Force

$$S_2 = -EI\frac{\mathrm{d}^3\bar{u}_2}{\mathrm{d}x_1^3} + N_1\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1} \tag{4}$$

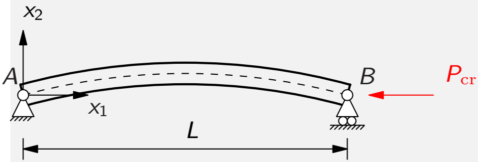
Buckling of Beams

Pre-buckling state



- Applied compressive load P is smaller than a critical load P_{cr}
- No lateral deflection of the column
- Column is in a state of stable equilibrium

Buckling



- Applied compressive load is equal to the critical load P_{cr}
- Lateral deflections occur
- Column is in a state of neutral equilibrium

Important questions

- At which critical load does "buckling" occur?
- What shape does the buckled column assume?
- Will buckling always occur provided that the applied compressive load is "large enough"?

Basic solution approach

- Insert $N_1 = -P$ (constant) and the assumption $p_2(x_1) = 0$ into the ODE (3) to obtain

$$\boxed{EI\frac{\mathrm{d}^4\bar{u}_2}{\mathrm{d}x_1^4} + P\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = 0} \tag{5}$$

- Note that Eq. (5) is in fact a (continuous and generalized) eigenproblem of the form $A[\bar{u}_2] + \lambda B[\bar{u}_2] = 0$ (where A and B are known linear operators, and λ is an unknown scalar). Its non-trivial solutions (where $\bar{u}_2(x_1) \neq 0$) comprise the unknown eigenvalues P and the corresponding eigenvectors $\bar{u}_2(x_1)$. The latter are also called mode shapes and actually functions for continuous eigenproblems, not vectors.
- Solve Eq. (5) for the unknown eigenvalues P and the corresponding eigenvectors $\bar{u}_2(x_1)$ subject to the desired boundary conditions of the column. Since Eq. (5) is a fourth-order ODE, four boundary conditions will be required.

General solution

- Now, divide Eq. (5) by EI:

$$\boxed{\frac{\mathrm{d}^4 \bar{u}_2}{\mathrm{d}x_1^4} + \frac{P}{EI}\frac{\mathrm{d}^2 \bar{u}_2}{\mathrm{d}x_1^2} = 0} \tag{6}$$

- The ansatz $\bar{u}_2(x_1) = \mathrm{e}^{\lambda x_1}$ then yields the characteristic polynomial

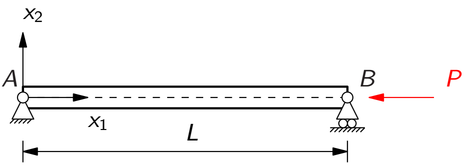
$$\lambda^4 + \frac{P}{EI}\lambda^2 = 0$$

which has the roots $\lambda_{1,2} = 0$ and $\lambda_{3,4} = \pm \mathrm{i}\sqrt{\frac{P}{EI}}$.

- Consequently, the general solution to Eq. (6) is

$$\boxed{\bar{u}_2(x_1) = A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B \cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) + C\, x_1 + D} \tag{7}$$

where the constants A, B, C, D must be determined from boundary conditions.



Boundary conditions:

$$\begin{aligned}
\bar{u}_2(x_1 = 0) &= 0\\
\bar{u}_2(x_1 = L) &= 0\\
M_3(x_1 = 0) &= EI\,\bar{u}_2''(x_1 = 0) = 0\\
M_3(x_1 = L) &= EI\,\bar{u}_2''(x_1 = L) = 0
\end{aligned}$$

Note:

$$\bar{u}_2''(x_1) = -\frac{P}{EI} A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) - \frac{P}{EI} B \cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) \tag{8}$$

$$\left.\begin{aligned}
\bar{u}_2(x_1 = 0) = 0 \quad &\Rightarrow \quad B + D = 0\\
\bar{u}_2''(x_1 = 0) = 0 \quad &\Rightarrow \quad B = 0
\end{aligned}\right\} \quad \boxed{B = D = 0} \tag{9}$$

$$\left.\begin{aligned}
\bar{u}_2(x_1 = L) = 0 \quad &\Rightarrow \quad A \sin\left(\sqrt{\tfrac{P}{EI}}\, L\right) + C\,L = 0\\
\bar{u}_2''(x_1 = L) = 0 \quad &\Rightarrow \quad A \sin\left(\sqrt{\tfrac{P}{EI}}\, L\right) = 0
\end{aligned}\right\} \quad \boxed{C = 0} \tag{10}$$

Remaining condition:

$$\boxed{A \sin\left(\sqrt{\frac{P}{EI}}\,L\right) = 0} \tag{11}$$

Fulfilled if

- $A = 0$
  (Then $A = B = C = D = 0$ which yields the trivial solution $\bar{u}_2(x_1) = 0$.)

or if

- $\boxed{\sqrt{\frac{P}{EI}}\,L = n\pi}$ where n is any integer.

Thus, buckling occurs for a simply-supported column if:

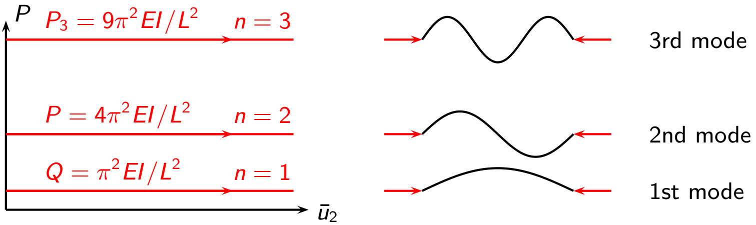
$$\boxed{P = \frac{n^2\pi^2 EI}{L^2} \quad \text{for } n \in \mathbb{N}^+} \qquad \text{eigenvalues} \tag{12}$$

For each value of P, there is an associated deflection solution (mode shape):

$$\boxed{\bar{u}_2(x_1) = A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) = A \sin\left(\frac{n\pi x_1}{L}\right)} \qquad \text{eigenvectors/mode shapes} \tag{13}$$

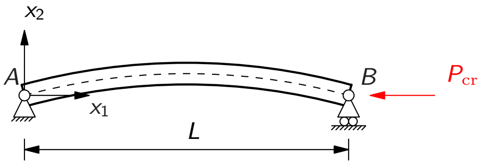
Note that the above deflections are solutions for any value of A. This corresponds to the well-known result that the eigenvectors/mode shapes in an eigenproblem can be only determined up to an arbitrary (non-zero) constant.

Consider the buckling loads and associated mode shapes for $n = 1, 2, 3$:



The lowest buckling load (obtained for $n = 1$) is the one for which buckling occurs:

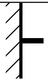
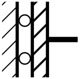
$$\boxed{P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2}}$$



Note: Higher critical loads and their associated mode shapes can be reached if the column is "artificially restrained" at lower bifurcation loads.

Buckling of Beams I

Homogeneous boundary conditions at the ends of the column (same as for beam bending):

Name	Symbol	Boundary condition
Fixed, built-in or clamp support		$\bar{u}_2 = 0$ $\bar{u}_2' = 0$
Roller		$\bar{u}_2 = 0$ $M_3 = EI\,\bar{u}_2'' = 0$
Clamp on a roller		$\bar{u}_2' = 0$ $S_2 = -EI\,\bar{u}_2''' - P\bar{u}_2' = 0$
Free end		$M_3 = EI\,\bar{u}_2'' = 0$ $S_2 = -EI\,\bar{u}_2''' - P\bar{u}_2' = 0$

General solution procedure:

- Start with the general solution to the fourth-order ODE:

$$\boxed{\bar{u}_2(x_1) = A\sin\left(\sqrt{\frac{P}{EI}}\,x_1\right) + B\cos\left(\sqrt{\frac{P}{EI}}x_1\right) + Cx_1 + D} \tag{14}$$

- Apply four boundary conditions (two at each end) to determine A, B, C, D. In general, this yields a homogeneous linear system of equations:

$$\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{4\times4 \text{ matrix M}} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

- In order to have a non-trivial solution to A, B, C, D, the determinant of M has to be zero. Thus, set $\det(\mathrm{M}) = 0$, and determine the roots P_{cr} of the resulting equation. These roots correspond to the eigenvalues.

- For homogeneous boundary conditions, the critical buckling load has the generic form:

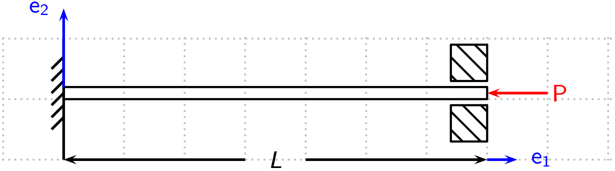
$$\boxed{P_{\mathrm{cr}} = \frac{c\pi^2 EI}{L^2}}$$

Here, c is the coefficient of edge fixity. It depends on the considered combination of boundary conditions.

- Since we are solving an eigenproblem, there is no unique solution to A, B, C, D. They can be determined up to some constant. Inserting an eigenvalue P_{cr} and the associated values for A, B, C, D into the general solution to $\bar{u}_2(x_1)$ yields the mode shape corresponding to P_{cr}.

Buckling of a uniform beam clamped at both ends

Consider the case of a uniform beam clamped at both ends and loaded by a uniform axial force P at $(x_1 = L)$ which acts on the beam neutral axis.



The boundary conditions are:

$$\begin{cases} u_2(x_1 = 0) = 0 \\ u_2'(x_1 = 0) = 0 \end{cases} \text{ and } \begin{cases} u_2(x_1 = L) = 0 \\ u_2'(x_1 = L) = 0 \end{cases} \tag{16}$$

The goal again is to use the boundary conditions to determine values of the load P for which the beam can be in equilibrium in a deformed configuration, (i.e. we have a non-trivial solution $\bar{u}_2 \neq 0$). We obtain:

$$
\begin{aligned}
\bar{u}_2(x_1 = 0) &= A\sin(\sqrt{\frac{P}{EI}} \times 0) + B\cos(\sqrt{\frac{P}{EI}} \times 0) + C \times 0 + D \\
&= \boxed{B + D = 0} \\
\bar{u}_2'(x_1 = 0) &= \sqrt{\frac{P}{EI}} A\cos(\sqrt{\frac{P}{EI}} \times 0) - \sqrt{\frac{P}{EI}} B\sin(\sqrt{\frac{P}{EI}} \times 0) + C \\
&= \boxed{\sqrt{\frac{P}{EI}} A + C = 0} \\
\bar{u}_2(x_1 = L) &= \boxed{A\sin(\sqrt{\frac{P}{EI}} \times L) + B\cos(\sqrt{\frac{P}{EI}} \times L) + C \times L + D = 0} \\
\bar{u}_2'(x_1 = L) &= \boxed{\sqrt{\frac{P}{EI}} A\cos(\sqrt{\frac{P}{EI}} \times L) - \sqrt{\frac{P}{EI}} B\sin(\sqrt{\frac{P}{EI}} \times L) + C = 0}
\end{aligned}
$$

We obtain the following system:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sqrt{\frac{P}{EI}} & 0 & 1 & 0 \\ \sin(\sqrt{\frac{P}{EI}} \times L) & \cos(\sqrt{\frac{P}{EI}} \times L) & L & 1 \\ \sqrt{\frac{P}{EI}} \cos(\sqrt{\frac{P}{EI}} \times L) & -\sqrt{\frac{P}{EI}} \sin(\sqrt{\frac{P}{EI}} \times L) & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For a non trivial solution:

$$\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

the matrix must be singular, i.e. its determinant must vanish. Let's call this matrix H and define $k = \sqrt{\frac{P}{EI}}$.

$$
\begin{aligned}
\det(H) &= -\begin{vmatrix} k & 1 & 0 \\ \sin(kL) & L & 1 \\ k\cos(kL) & 1 & 0 \end{vmatrix} - \begin{vmatrix} k & 0 & 1 \\ \sin(kL) & \cos(kL) & L \\ k\cos(kL) & -k\sin(kL) & 1 \end{vmatrix} \\
&= +\begin{vmatrix} k & 1 \\ k\cos(kL) & 1 \end{vmatrix} - k\begin{vmatrix} \cos(kL) & L \\ -k\sin(kL) & 1 \end{vmatrix} - \begin{vmatrix} \sin(kL) & \cos(kL) \\ k\cos(kL) & -k\sin(kL) \end{vmatrix} \\
&= k(1-\cos(kL)) - k(\cos(kL) + Lk\sin(kL)) + k\sin^2(kL) + k\cos^2(kL) \\
&= k - k\cos(kL) - k\cos(kL) - Lk^2\sin(kL) + k \\
&= 2k - 2k\cos(kL) - Lk^2\sin(kL) \\
&= 2k\left(1 - \cos(kL) - \frac{kL}{2}\sin(kL)\right) \\
&= 2k\left(1 - [1 - 2\sin^2(\frac{kL}{2})] - \frac{kL}{2} \times 2\sin(\frac{kL}{2})\cos(\frac{kL}{2})\right) \\
&= 4k\sin(\frac{kL}{2})\left(\sin(\frac{kL}{2}) - \frac{kL}{2}\cos(\frac{kL}{2})\right) = 0
\end{aligned}
$$

The equilibrium in the deformed configuration can then be satisfied if any of these conditions are held:

1. $k = 0$
2. $\sin(\frac{kL}{2}) = 0$, or
$$k = \frac{2n\pi}{L} \tag{17}$$
3. $\sin(\frac{kL}{2}) - \frac{kL}{2}\cos(\frac{kL}{2}) = 0$, or
$$\tan(\frac{kL}{2}) = \frac{kL}{2} \; . \tag{18}$$

In Case (17), we can replace k by its original expression and obtain

$$\sqrt{\frac{P}{EI}} = \frac{2n\pi}{L}$$

hence:

$$\boxed{P_{cr}^{n} = \frac{4n^2\pi^2 EI}{L^2}}$$

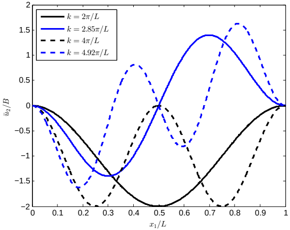
In this case, the displacement mode shape function $\bar{u}_2$ becomes:

$$\bar{u}_2 = B\left(\cos(kx_1) - 1\right) = B\left(\cos(\frac{2n\pi}{L}x_1) - 1\right)$$

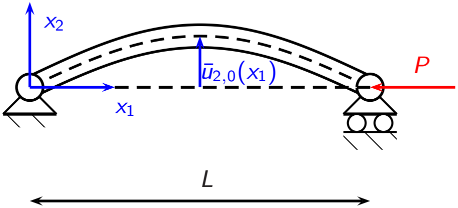
In Case (18), the solution is also a series of numbers due to the periodicity of tangential function. The solutions can be obtained numerically, and the first two are $\frac{8.97}{L}(=\frac{2.85\pi}{L})$, $\frac{15.45}{L}(=\frac{4.92\pi}{L})$, which lead to

$$P_{cr} = k^2 EI = \frac{80.76EI}{L^2}(=\frac{8.18\pi^2 EI}{L^2}),\ \frac{238.72EI}{L^2}(=\frac{24.19\pi^2 EI}{L^2}), \ldots$$

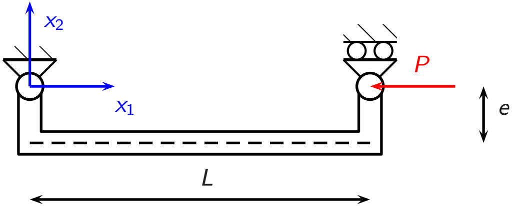
Set $B = 1$ and then coefficients A, C, D can be determined by solving the reduced linear system, see Mathematica notebook attached. The first two deformation modes from Case (17) and Case (18) are plotted in the figure



- Type 1: Initial deflection in the column (due to manufacturing, etc.)



- Type 2: Load not applied along centerline of column



Solution approach

- ODE governing the deflection $\bar{u}_2$:

$$\boxed{\frac{\mathrm{d}^4\bar{u}_2}{\mathrm{d}x_1^4} + \frac{P}{EI}\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = 0}$$

- Consider general solution from before:

$$\boxed{\bar{u}_2(x_1) = A\sin\left(\sqrt{\frac{P}{EI}}\,x_1\right) + B\cos\left(\sqrt{\frac{P}{EI}}\,x_1\right) + C\,x_1 + D}$$

- Boundary conditions for Type 2:

$$\bar{u}_2(x_1 = 0) = 0 \qquad M_3(x_1 = 0) = EI\,\bar{u}_2''(x_1 = 0) = Pe$$
$$\bar{u}_2(x_1 = L) = 0 \qquad M_3(x_1 = L) = EI\,\bar{u}_2''(x_1 = L) = Pe$$

Integration constants for Type 2:

$$\left.\begin{array}{lcc} \bar{u}_2(x_1=0)=0 & \Rightarrow & B+D=0 \\ \bar{u}_2''(x_1=0)=\frac{Pe}{EI} & \Rightarrow & B=-e \end{array}\right\} \quad \boxed{B=-e,\;\; D=e} \tag{19}$$

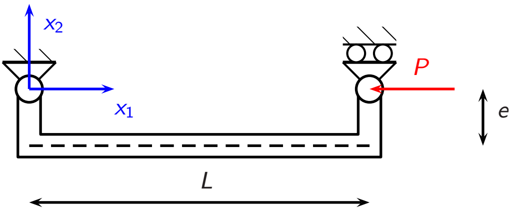
$$\underbrace{\begin{array}{lcr} \bar{u}_2(x_1=L)=0 & \Rightarrow & A\sin\left(\sqrt{\frac{P}{EI}}\,L\right)+C\,L=-e\left[1-\cos\left(\sqrt{\frac{P}{EI}}\,L\right)\right] \\ \bar{u}_2''(x_1=L)=\frac{Pe}{EI} & \Rightarrow & A\sin\left(\sqrt{\frac{P}{EI}}\,L\right)=-e\left[1-\cos\left(\sqrt{\frac{P}{EI}}\,L\right)\right] \end{array}}_{\boxed{A=-e\,\frac{1-\cos\left(\sqrt{\frac{P}{EI}}\,L\right)}{\sin\left(\sqrt{\frac{P}{EI}}\,L\right)},\quad C=0}} \tag{20}$$

Complete solution for Type 2:

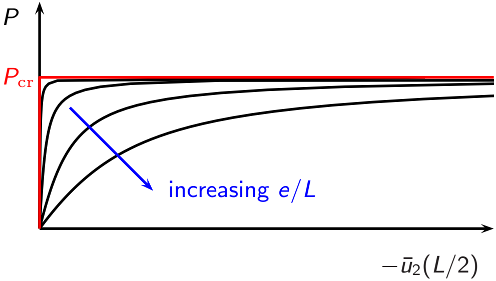
$$\boxed{\bar{u}_2(x_1) = -e\left[\frac{1-\cos\left(\sqrt{\frac{P}{EI}}\,L\right)}{\sin\left(\sqrt{\frac{P}{EI}}\,L\right)}\sin\left(\sqrt{\frac{P}{EI}}\,x_1\right)+\cos\left(\sqrt{\frac{P}{EI}}\,x_1\right)-1\right]} \tag{21}$$

Note:

- All constants A, B, C, D determined uniquely for $P \neq \frac{n^2\pi^2 EI}{L^2}$.
- $\bar{u}_2(x_1)$ is finite for $P \neq \frac{n^2\pi^2 EI}{L^2}$.
- $\bar{u}_2(x_1)$ unbounded in the limit $P \to \frac{n^2\pi^2 EI}{L^2}$.



Response of column to eccentric load



- Bifurcation is asymptote
- $\bar{u}_2$ approaches bifurcation as $P \to P_{\mathrm{cr}}$
- As e/L (imperfection) increases, behavior deviates from perfect case (bifurcation)

Failure of columns

Clearly, in the "perfect" case, a column will fail if it buckles:

$$\bar{u}_2 \to \infty \quad \text{(not very useful)}$$

$$\bar{u}_2 \to \infty \quad \Rightarrow \quad M_3 \to \infty \quad \Rightarrow \quad \sigma_{11} \to \infty \quad \Rightarrow \quad \text{material fails!}$$

Buckling of Beams

Let's consider what else could happen depending on geometry:

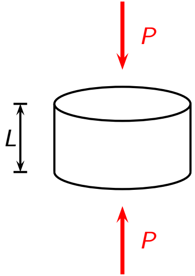
- For a long and slender column:

$$P_{\mathrm{cr}} = \frac{c\pi^2 EI}{L^2}$$

Using $\sigma_{11} = -P/A$: $\boxed{\sigma_{\mathrm{cr}} = \frac{c\pi^2 EI}{L^2 A}}$ buckling stress (compressive)

- For a short column:
  If no buckling occurs, the column will fail when the ultimate compressive stress σ_{cu} of the material is reached:

$$\sigma_{11} = -\frac{P}{A} = -\sigma_{\mathrm{cu}}$$



$\Rightarrow$ Failure by "squashing"

Behavior of columns of various geometries:

Characteristic quantities:

- Effective length: $\tilde{L} = L/\sqrt{c}$ (depends on boundary conditions)
- Radius of gyration: $\rho = \sqrt{I/A}$ (ratio of moment of inertia to area)

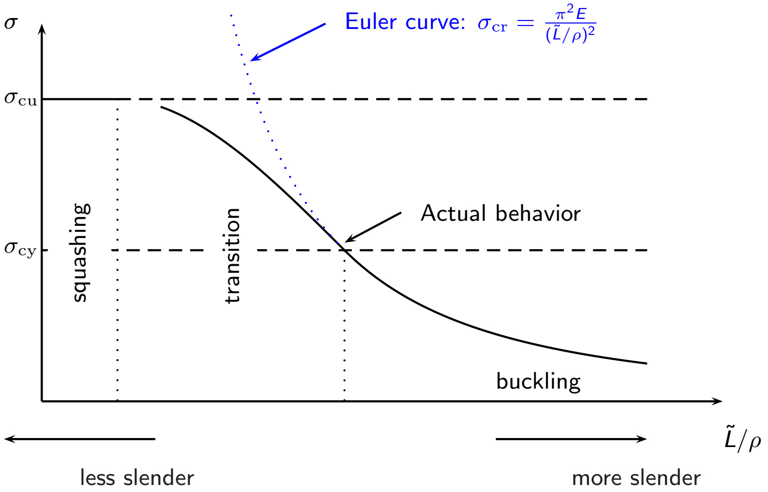
Looking at $\sigma_{\mathrm{cr}} = \frac{c\pi^2 EI}{L^2 A}$, one finds:

$$\boxed{\sigma_{\mathrm{cr}} = \frac{c\pi^2 E}{\left(\tilde{L}/\rho\right)^2}}$$

One can capture the behavior of columns of various geometries on one plot using:

- Slenderness ratio: $\tilde{L}/\rho$

General behavior of columns for various slenderness ratios:



Notes:

- For $\tilde{L}/\rho$ "large", the column fails by buckling.
- For $\tilde{L}/\rho$ "small", the column squashes.
- In the transition region, plastic deformation (yielding) takes place:

$$\sigma_{\text{cy}} \ < \ |\sigma_{11}| \ < \ \sigma_{\text{cu}}$$

- σ_{cy}: compressive yield stress
- σ_{cu}: ultimate compressive stress