Transverse loading of slender structural elements: Beam Theory

Reading: CDL Chap. 3, 7 and 8

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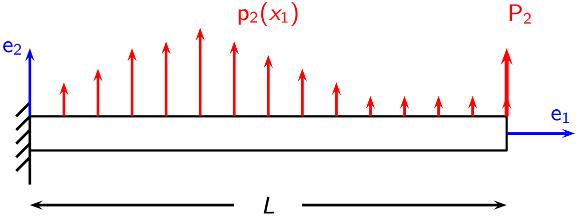
Outline



Beams have the defining characteristic that they can resist loads acting transversely to its axis by bending or deflecting orthogonally to their axis. This bending deformation causes internal axial and shear stresses which can be described by equipolent stress resultant moments and shearing forces.

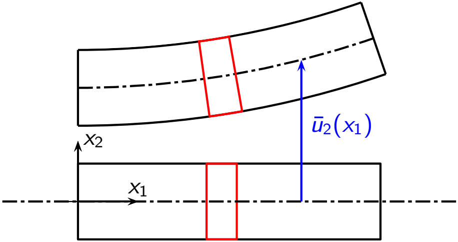
Our goal is to compute the internal resultant forces and moments, stresses and deformations of a beam subjected to general loading, as shown in the figure.

This includes, applied concentrated loads and moments, and distributed forces.

Kinematic assumption:

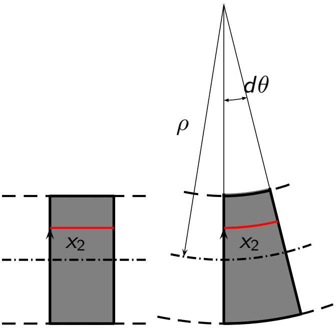
Beam theory is founded on the following two key assumptions known as the Euler-Bernoulli assumptions:

- Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid
- During deformation, the cross sections of the beam are assumed to remain planar and normal to the deformed axis of the beam.
- This implies that locally lines perpendicular to the axis of the beam are the radii of a circle of radius equal to the local radius of curvature ρ of the curve described by $\bar{u}_2(x_1)$ (which is not necessarily a circle)



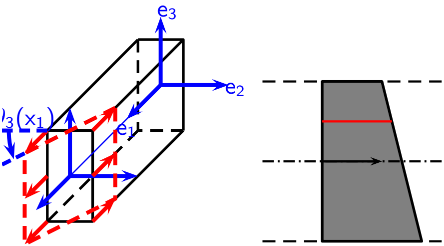
The red segment in the figure is at a distance x_2 from the beam fiber which does not deform (neutral axis). It's length is initially dx_1. In the deformed beam, the neutral axis still has a length $ds = dx_1$ which is now also $\rho d\theta$. The new length of the red segment is $ds' = (\rho - x_2)d\theta$. The change in length is $ds' - ds = (\cancel{\rho} - x_2 - \cancel{\rho})d\theta$. And the longitudinal strain is then:

$$\varepsilon_{11} = \frac{-x_2}{\rho}$$



From Calculus, the radius of curvature of a function $y(x)$ is given by $\rho(y(x)) = \frac{(1+y'(x)^2)^{\frac{3}{2}}}{y''(x)} \Rightarrow$

$$\varepsilon_{11} = \frac{-x_2 \bar{u}_2''(x_1)}{(1+\bar{u}_2'(x_1)^2)^{\frac{3}{2}}} \sim -x_2 \bar{u}_2''(x_1)$$



Transverse deflections:

$$\boxed{u_2(x_1, x_2, x_3) = \bar{u}_2(x_1)}$$

The kinematic restrictions imposed by the second Euler-Bernoulli assumption results in the following form of the u_1 displacement component:

$$u_1(x_1, x_2, x_3) = -x_2\theta_3(x_1), \; \theta_3 = \frac{d\,\bar{u}_2}{dx_1}$$

Summarizing:

Kinematic assumption

$$u_1(x_1, x_2, x_3) = -x_2 \frac{d\bar{u}_2}{dx_1} \tag{1}$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1) \tag{2}$$

Strain field: Using the kinematic assumptions of Euler-Bernoulli beam theory we can derive the general form of the strain field:

$$\boxed{\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = -x_2 \bar{u}_2''(x_1)} \tag{3}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{d\,\bar{u}_2(x_1)}{dx_2} = 0 \tag{4}$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{d}{dx_2}\left(-x_2 \bar{u}_2'(x_1)\right) + \bar{u}_2'(x_1) = -\underbrace{\frac{dx_2}{dx_2}}_{1} \bar{u}_2'(x_1) + \bar{u}_2'(x_1) = 0, \tag{5}$$

These expressions give us a deeper insight on the nature of the strains due to bending resulting from the Euler-Bernoulli assumption on the deformation of beams:

- The expression for ε_{11} tells us that the axial fibers of the beam stretch and contract proportionally to the distance to the plane $x_1 - x_3$, where we have assumed that the fibers have no stretch. We will call this the neutral axis.

- The constant of proportionality is the second derivative of the function describing the deflections of the axis of the beam. This can be seen as a linearized version of the local value of the curvature.
- There are no shear strains!!!! This is a direct consequence of assuming that the cross-section remains normal to the deformed axis of the beam.
- There are no strains in the plane. This is a direct consequence of assuming that the cross section is rigid.

One of the main conclusions of the Euler-Bernoulli assumptions is that in this particular beam theory the primary unknown variable is the deflection function $\bar{u}_2(x_1)$ which is only a function of x_1. The full displacement, strain and therefore stress fields do depend on the other independent variables but in a prescribed way that follows directly from the kinematic assumptions and from the equations of elasticity. The purpose of formulating a beam theory is to obtain a description of the problem expressed entirely on variables that depend on a single independent spatial variable x_1 which is the coordinate along the axis of the beam.

Constitutive law for the cross section We will assume a linear-elastic isotropic material and that the transverse stresses $\sigma_{22}, \sigma_{33} \sim 0$. By Hooke's law, the axial stress σ_{11} is given by:

$$\sigma_{11}(x_1, x_2, x_3) = E\varepsilon_{11}(x_1, x_2) = -Ex_2\bar{u}_2''(x_1) \tag{6}$$

In other words, we are assuming a state of uni-axial stress.
This exposes an inconsistency in Euler-Bernoulli beam theory: we are assuming the kinematics to be uni-axial strain, and the kinetics to be uni-axial stress. In other words one can either have:

$$\varepsilon_{22} = \varepsilon_{33} = 0$$

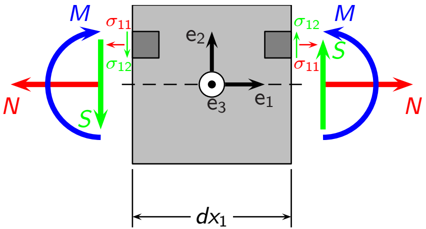
(Euler-Bernoulli hypothesis) or

$$\sigma_{22} = \sigma_{33} = 0$$

These two cannot co-exist except when the Poisson ratio is zero. However, the inconsistency in general has a small effect in most problems of practical interest.

The theory is developed assuming that we can ignore both these strains and stresses.
Assuming E is constant in the cross section, it can be seen from Equation (6) that the σ_{11} stress distribution through the thickness is linear in x_2.

Stress resultants: are equivalent force systems that represent the integral effect of the internal stresses acting on the cross section. Thus, they eliminate the need to carry over the dependency of the stresses on the spatial coordinates of the cross section x_2, x_3.



We've already discussed the *axial or normal force*:

$$\boxed{N(x_1) = \int_A \sigma_{11}(x_1, x_2, x_3) dA} \tag{7}$$

The *bending moment* is defined by the expression:

$$\boxed{M(x_1) = -\int_A x_2 \sigma_{11}(x_1, x_2) dA} \tag{8}$$

The negative sign is needed to generate a positive bending moment with respect to axis e_3, as shown in the figure.
The *transverse shear force* is defined by the expression:

$$\boxed{S(x_1) = \int_A \sigma_{12}(x_1, x_2) dA} \tag{9}$$

Derivation of the moment-curvature relation for beams

Specialize the definition of the M stress resultant (8)

$$M(x_1) = -\int_{A(x_1)} x_2 \sigma_{11}(x_1, x_2) dA$$

to the case under consideration by using the stress distribution resulting from the Euler-Bernoulli hypothesis, $\sigma_{11}(x_1, x_2, x_3) = -E x_2 \bar{u}_2''(x_1)$ to obtain a relation between the bending moment and the local curvature $\bar{u}_2''(x_1)$:

$$M(x_1) = \cancel{-}\int_{A(x_1)} x_2 \cancel{(-1)} E x_2 \bar{u}_2''(x_1) dA = \underbrace{\left[\int_{A(x_1)} E x_2^2 dA\right]}_{H(x_1)} \bar{u}_2''(x_1)$$

We can see that we obtain a linear relation between the bending moment and the local curvature (*moment-curvature relationship*):

$$\boxed{M_3(x_1) = H(x_1) \bar{u}_2''(x_1)} \qquad (10)$$

The constant of proportionality will be referred to as the *bending stiffness* (also sometimes known as the *flexural rigidity*):

$$\boxed{H(x_1) = \int_{A(x_1)} E x_2^2 \, dA} \tag{11}$$

In the case of a homogeneous cross section of Young's modulus $E(x_1)$:

$$H(x_1) = E(x_1) \underbrace{\int_{A(x_1)} x_2^2 \, dA}_{I} \tag{12}$$

we obtain:

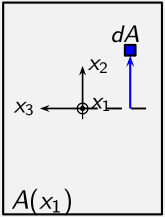
$$\boxed{M = EI \bar{u}_2''(x_1)} \tag{13}$$

The quantity I in Equation (12) is defined as the Moment of inertia of the area of the cross section:

Definition

Moment of inertia of the cross section

$$\boxed{I = \int_{A(x_1)} x_2^2 \, dA} \qquad (14)$$

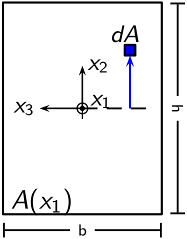


Example

Moment of inertia of a rectangular beam of base b and height h

$$I = \int_{A(x_1)} x_2^2 dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} x_2^2 dx_3 dx_2 =$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_2^2 \underbrace{\left(\int_{-\frac{b}{2}}^{\frac{b}{2}} dx_3 \right)}_{b} dx_2 = b \frac{x_2^3}{3} \Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \boxed{I = \frac{bh^3}{12}}$$



An important consequence of the moment-curvature relation (13) is that the stress distribution in the beam can be expressed in terms of the internal moment. To see this, solve for $u''(x_1)$ in this equation and replace in (6), and obtain:

$$\sigma_{11}(x_1, x_2) = -\cancel{E} \underbrace{\frac{M(x_1)}{\cancel{E}I}}_{u''(x_1)} x_2 \tag{15}$$

$$\boxed{\sigma_{11}(x_1, x_2) = -\frac{M(x_1)}{I} x_2} \tag{16}$$

Extremum (tensile and compressive) stresses are found at the top and bottom edges of the beam and in the cross section x_1 where the bending moment is maximum.

Example

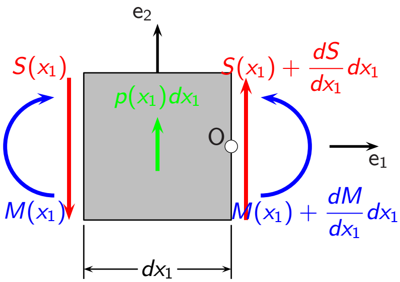
Stresses in a rectangular beam: In this case, $I = \frac{bh^3}{12}$, then:

$$\sigma_{11}(x_1, x_2) = -\frac{12M(x_1)}{bh^3}x_2$$

$$\sigma_{11}^{min,max}(x_1) = -\frac{\cancelto{6}{12}M(x_1)}{bh^{\cancelto{2}{3}}}\frac{\pm\cancelto{1}{h}}{\cancelto{1}{2}} = \mp\frac{6M(x_1)}{bh^2}$$

Differential equations of Equilibrium for beams

Principle of equilibrium We will formulate the internal equilibrium of beams by analyzing the free body diagram (FBD) of a differential element of the beam along the beam axis dx_1. The forces and moments acting on the beam element are: the internal resultant shear force distribution $S(x_1)$, the internal bending moment distribution $M(x_1)$, and the distributed force per unit length $p(x_1)$.



Equilibrium of forces in the e_2 direction gives: $\cancel{-S(x_1)} + p_2(x_1)dx_1 + \cancel{S(x_1)} + S'(x_1)dx_1 = 0, \Rightarrow \boxed{S'(x_1) + p_2(x_1) = 0}$

Equilibrium of moments about point **O** gives: $\cancel{-M(x_1)} + S(x_1)dx_1 - \cancelto{h.o.t.}{p_2(x_1)dx_1\frac{dx_1}{2}} + \cancel{M(x_1)} + M'(x_1)dx_1 = 0, \Rightarrow \boxed{M'(x_1) + S(x_1) = 0}$

These two equations can be combined by differentiating the second equation: $\left(M'(x_1) + S(x_1)\right)' = M''(x_1) + S'(x_1) = 0$, and replacing in the first one: $-M''(x_1) + p_2(x_1) = 0$

$$\boxed{M''(x_1) = p_2(x_1)}$$

In summary, equilibrium of beams poses the following restrictions on the distribution of bending moments $M(x_1)$ and shear forces $S(x_1)$.

Beam equilibrium equations

$$\boxed{S'(x_1) + p_2(x_1) = 0} \tag{17}$$

$$\boxed{M'(x_1) + S(x_1) = 0} \tag{18}$$

$$\boxed{M''(x_1) = p_2(x_1)} \tag{19}$$

Governing equation for beam deflections

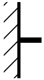
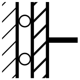
By combining Equations (13) with the moment equilibrium equation (19).

$$\boxed{\left(EI(x_1)\bar{u}_2''(x_1)\right)'' = p(x_1)} \tag{20}$$

Boundary conditions for the equilibrium equation: As any fourth-order ODE, the governing equation for beams (20) requires four boundary conditions (typically at the beam ends). From ODE theory, we know that these BCs can be given in terms of the unknown function $\bar{u}_2(x_1)$ or any of its derivatives up to the third order (one order less than the ODE). The following table describes the types of boundary conditions that we can find in beams:

Type of BC	Value constrained	Can't impose at the same time
Deflection	$\bar{u}_2(x^*)$	Shear $S(x^*) = -EI\,\bar{u}_2'''$
Rotation	$\bar{u}_2'(x^*)$	Moment $M(x^*) = EI\,\bar{u}_2''$
Moment	$EI\,\bar{u}_2''(x^*)$	Rotation $\bar{u}_2'$
Shear	$-EI\,\bar{u}_2'''(x^*)$	Deflection $\bar{u}_2$

The symbols we use to represent practical types of supports are as we've discussed before:

Name	Symbol	Constrained dof	Unknown
Fixed, built-in or clamp support		Deflection $\bar{u}_2$ and rotation $\bar{u}_2'$	Shear S, Moment M
Roller		Deflection $\bar{u}_2$, Moment M	Shear S (Force Reaction), u_2' (rotation)
Clamp on a roller		Rotation $\bar{u}_2'$, Shear S	Moment M, deflection $\bar{u}_2$
Free end		Moment M, Shear S	Deflection $\bar{u}_2$, rotation $\bar{u}_2'$

Shear stresses in beams subject to bending: The equilibrium equations tell us that bending moments cannot vary along the beams unless there are non-zero internal shear forces ($M' + S = 0$), and that distributed forces will produce variations in the internal shear force distribution along the beam ($S' + p_2 = 0$). In other words, internal shear forces S, not only exist, but they also play an integral role in the theory. However, our derivation of the theory showed that one of the consequences of the kinematic Euler-Bernoulli hypothesis is that there are no distortions and therefore no shear stresses in this theory.

Here, we will show that stress equilibrium not only demands the presence of shear stresses, it also gives the mathematical framework to compute their distribution. Start from the first equation of equilibrium for stresses (no body forces):

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

and replace the known form of $\sigma_{11}(x_1, x_2) = -\frac{M(x_1)}{I} x_2$, equation (16) from beam theory from above:

$$\frac{\partial}{\partial x_1}\left(-\frac{M(x_1)}{I} x_2\right) + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_2} = \frac{\overbrace{M'(x_1)}^{-S(x_1)}}{I} x_2$$

$$\sigma_{12} = -\frac{S(x_1)}{I}\frac{x_2^2}{2} + C$$

The boundary condition is found at the top and bottom surfaces of the beam, where the shear stress is zero. In the particular case of a beam of rectangular cross section, this corresponds to $x_2 = \pm\frac{h}{2}$, giving:

$$C = \frac{S(x_1)}{I}\frac{\left(\frac{h}{2}\right)^2}{2}$$

$$\boxed{\sigma_{12}(x_1, x_2) = \frac{S(x_1)}{2I}\left[\left(\frac{h}{2}\right)^2 - x_2^2\right]}$$

It is found that shear stresses have a parabolic distribution in the cross section with the maximum at $x_2 = 0$ and decreasing to zero on both top and bottom surfaces.

To gain some perspective on the importance of shear vs normal stresses in beam bending, let's compare the following example.

Consider the simple situation of a rectangular cantilever beam subject to a concentrated load P on the free end. The shear distribution is constant $S(x_1) = P$, whereas the moment distribution is $M(x_1) = P(L - x_1)$. The maximum bending stresses then occur at $x_1 = 0$ and their absolute value is:

$$\max(|\sigma_{11}|) = \frac{\overbrace{PL}^{M(0)}}{I}\frac{h}{2}$$

The maximum shear stresses happen at $x_2 = 0$ (any x_1):

$$\max(\sigma_{12}) = \frac{\overbrace{P}^{S(x_1)}}{2I}\left(\frac{h}{2}\right)^2$$

Their ratio is:

$$\frac{\max(\sigma_{12})}{\max(|\sigma_{11}|)} = \frac{\frac{\cancel{P}}{2\cancel{I}}\left(\frac{h}{2}\right)^{\cancel{2}}}{\frac{\cancel{P}L}{\cancel{I}}\frac{\cancel{h}}{\cancel{2}}} = \frac{1}{4}\frac{h}{L}$$

For other beam configurations, the factor $1/4$ will change, but the overall scaling relation h/L persists. In many practical situations, beams are very long and slender, the factor h/L is very small, and so are the shear stresses compared to the dominant longitudinal stresses that support the bending moments. However, for short beams, shear stresses could be significant.