16.001 Unified Engineering
Materials and Structures

Summary of Equations of Elasticity

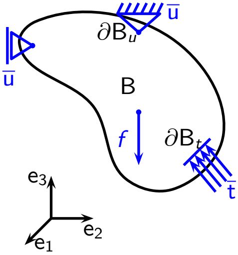
Reading assignments: CDL 5.6

Instructors: Raúl Radovitzky, Zachary Cordero
Teaching Assistants: Grégoire Chomette, Michelle Xu, and Daniel Pickard

Massachusetts Institute of Technology
Department of Aeronautics & Astronautics

Outline

Schematic of generic problem in linear elasticity

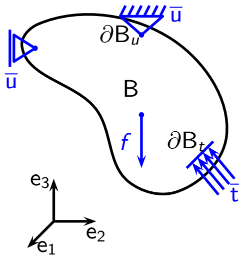


Three great principles in differential form

Equilibrium

$$\boxed{\sigma_{ji,j} + f_i = 0 \text{ in B}} \quad (1)$$

Schematic of generic problem in linear elasticity



Three great principles in differential form

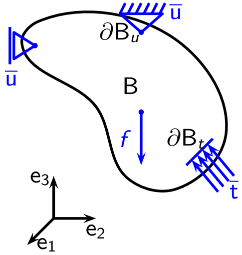
Equilibrium

$$\sigma_{ji,j} + f_i = 0 \text{ in B} \quad (1)$$

Compatibility

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ in B} \quad (2)$$

Schematic of generic problem in linear elasticity



Three great principles in differential form

Equilibrium

$$\sigma_{ji,j} + f_i = 0 \text{ in B} \quad (1)$$

Compatibility

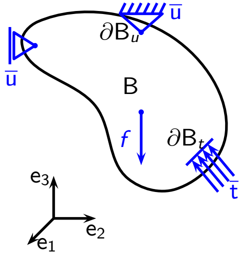
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ in B} \quad (2)$$

Stress-strain relations

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in B} \quad (3)$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \text{ in B} \quad (4)$$

Schematic of generic problem in linear elasticity



Three great principles in differential form

Equilibrium

$$\boxed{\sigma_{ji,j} + f_i = 0 \text{ in B}} \quad (1)$$

Compatibility

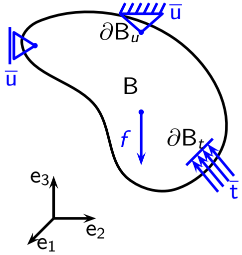
$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ in B}} \quad (2)$$

Stress-strain relations

$$\boxed{\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in B}} \quad (3)$$

$$\boxed{\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \text{ in B}} \quad (4)$$

Schematic of generic problem in linear elasticity



Equation, unknown count?

Three great principles in differential form

Equilibrium

$$\sigma_{ji,j} + f_i = 0 \text{ in B} \quad (1)$$

Compatibility

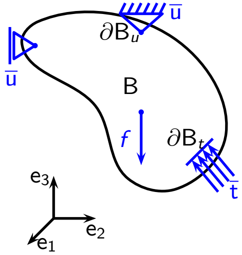
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ in B} \quad (2)$$

Stress-strain relations

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in B} \quad (3)$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \text{ in B} \quad (4)$$

Schematic of generic problem in linear elasticity



Equation, unknown count?

- Equilibrium: 3 equations, 6 unknowns

Three great principles in differential form

Equilibrium

$$\boxed{\sigma_{ji,j} + f_i = 0 \text{ in B}} \qquad (1)$$

Compatibility

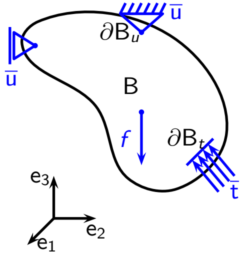
$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ in B}} \qquad (2)$$

Stress-strain relations

$$\boxed{\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in B}} \qquad (3)$$

$$\boxed{\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \text{ in B}} \qquad (4)$$

Schematic of generic problem in linear elasticity



Equation, unknown count?

- Equilibrium: 3 equations, 6 unknowns
- Compatibility: 6 equations, 9 unknowns

Three great principles in differential form

Equilibrium

$$\sigma_{ji,j} + f_i = 0 \text{ in B} \qquad (1)$$

Compatibility

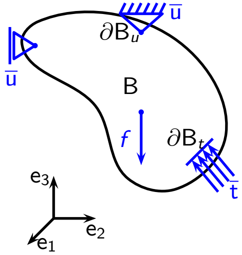
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ in B} \qquad (2)$$

Stress-strain relations

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in B} \qquad (3)$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \text{ in B} \qquad (4)$$

Schematic of generic problem in linear elasticity



Equation, unknown count?

- Equilibrium: 3 equations, 6 unknowns
- Compatibility: 6 equations, 9 unknowns
- Constitutive: 6 equations, 0 unknowns

Summary of equations of Elasticity: Stress-strain relations continued

For a linear isotropic elastic material:

- $$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$

For a linear isotropic elastic material:

- $$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$
- $$\begin{aligned}\varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\delta_{11}\right]\\ &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right]\end{aligned}$$

For a linear isotropic elastic material:

- $$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$
- $$\begin{aligned}\varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\delta_{11}\right] \\ &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right]\end{aligned}$$
- similarly for $\varepsilon_{22}, \varepsilon_{33}$

For a linear isotropic elastic material:

- $$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$
- $$\begin{aligned}\varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\delta_{11}\right] \\ &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right]\end{aligned}$$
- similarly for $\varepsilon_{22}, \varepsilon_{33}$
- $$\begin{aligned}\varepsilon_{23} &= \frac{1}{E}\left[(1+\nu)\sigma_{23} - \nu\sigma_{kk}\delta_{23}\right] \\ &= \frac{1+\nu}{E}\sigma_{23}\end{aligned}$$

For a linear isotropic elastic material:

- $$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$
- $$\begin{aligned}\varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\delta_{11}\right] \\ &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right]\end{aligned}$$
- similarly for $\varepsilon_{22}, \varepsilon_{33}$
- $$\begin{aligned}\varepsilon_{23} &= \frac{1}{E}\left[(1+\nu)\sigma_{23} - \nu\sigma_{kk}\delta_{23}\right] \\ &= \frac{1+\nu}{E}\sigma_{23}\end{aligned}$$
- similarly for $\varepsilon_{31}, \varepsilon_{12}$

Summary of equations of Elasticity

Stress-strain relations for linear elastic isotropic materials

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{33} + \sigma_{11}\right)\right]$$

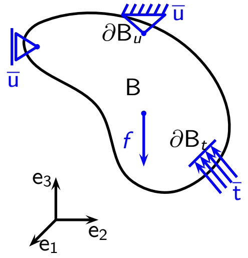
$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

$$2\varepsilon_{23} = \frac{1}{G}\sigma_{23}$$

$$2\varepsilon_{31} = \frac{1}{G}\sigma_{31}$$

$$2\varepsilon_{12} = \frac{1}{G}\sigma_{12}$$

Schematic of generic problem
in linear elasticity

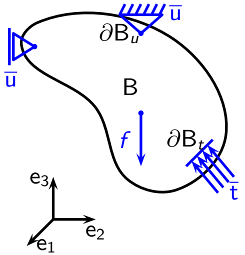


Equilibrium at the boundary: Traction or natural boundary conditions

For tractions $\bar{t}$ imposed on the portion of the surface of the body ∂B_t:

$$\boxed{n_i \sigma_{ij} = t_j = \bar{t}_j \text{ on } \partial B_t} \qquad (5)$$

Schematic of generic problem in linear elasticity



Summary of equations of Elasticity: Boundary conditions

Equilibrium at the boundary: Traction or natural boundary conditions

For tractions $\bar{t}$ imposed on the portion of the surface of the body ∂B_t:

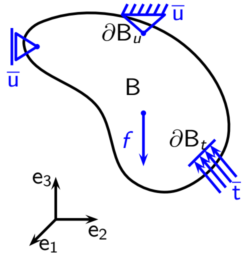
$$\boxed{n_i \sigma_{ij} = t_j = \bar{t}_j \text{ on } \partial B_t} \qquad (5)$$

Compatibility at the boundary: Displacement or essential boundary conditions

For displacements $\bar{u}$ imposed on the portion of the surface of the body ∂B_u, this includes the supports for which we have $\bar{u} = 0$

$$\boxed{u_i = \bar{u}_i} \qquad (6)$$

Schematic of generic problem in linear elasticity



Remark

We require $\partial B_u \cap \partial B_t = \emptyset$, since we cannot prescribe both traction and displacement at the same point!

We observe that the general elasticity problem contains 15 unknown fields: displacements (3), strains (6) and stresses (6); and 15 governing equations: equilibrium (3), pointwise compatibility (6), and constitutive (6), in addition to suitable displacement and traction boundary conditions. One can prove existence and uniqueness of the solution (the fields: $u_i(x_j)$, $\varepsilon_{ij}(x_k)$, $\sigma_{ij}(x_k)$) under some conditions on the elastic tensor (convexity of the strain energy function or the positive definiteness of the elastic tensor).
In the isotropic case, it can be shown that the system of equations has a solution (existence) which is unique (uniqueness) providing that the bulk and shear moduli are positive:

$$K = \frac{E}{3(1-2\nu)} > 0, \; G = \frac{E}{2(1+\nu)} > 0$$

which poses the following restrictions on the Poisson ratio:

$$-1 < \nu < 0.5$$

Exercise: write all 15 equations in expanded form