# 16.001 Unified Engineering
# Materials and Structures

## Introduction to Structural Analysis

Reading assignments: CDL 1.1, AJ Ch. 1

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## Introduction to Structural Analysis:

Structures can bear external loads by virtue of their ability to deform.

**Structural Analysis is concerned with the determination of :**

- deformation (external deflections and internal strains), and
- forces (reactions at supports and internal stresses)

that a structure experiences under applied loads

**Fundamental Principles**

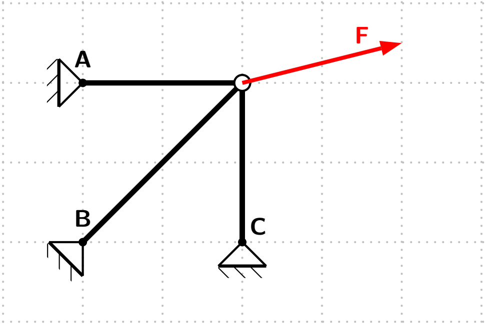
- Equilibrium
- Compatibility
- Constitutive relations

# Equilibrium

**Basic concept**

for a structure to remain in a static (albeit deformed) position, the loads applied on the structure and the reactions at the support must balance, i.e. there cannot be any resultant force or the structure would be accelerating.
Enforcing equilibrium gives conditions to determine internal forces in the structure.

Problem schematic



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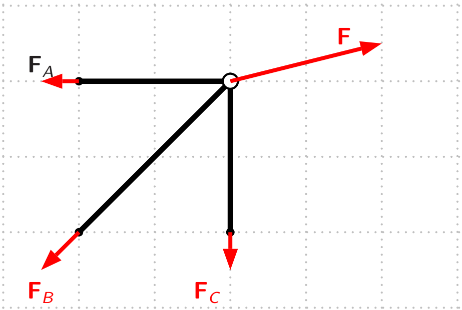
Free-body diagram



# Compatibility

## Basic concept

the structure must deform in a way that its internal connectivity is preserved. This means that joints must stay together, the structure must remain attached to the supports, new internal or external material surfaces should not appear and material points cannot overlap as a result of the deformation. No holes of overlaps (material interpenetration) can appear as a result of the deformation. Enforcing compatibility gives conditions to determine structure deformation.

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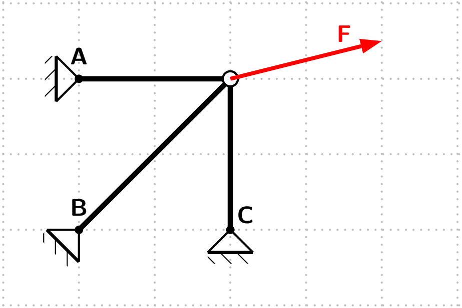

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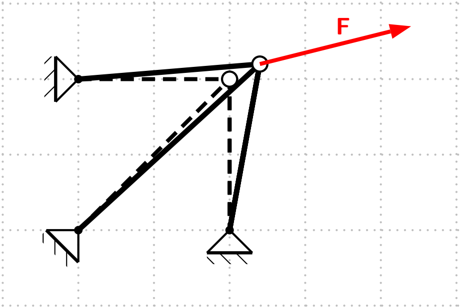
Compatible deformed state



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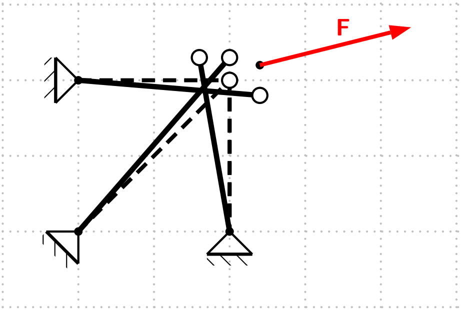
Incompatible deformed state

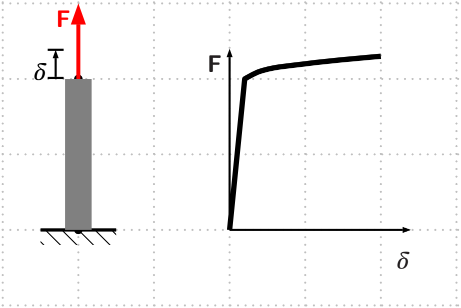


# Constitutive Material Response

## Basic concept

Determining internal forces in structure requires a knowledge of material behavior under load: How stiff is the material, i.e. how much does the material deform for a given load? Usually determined experimentally.

Material test



# Equilibrium:

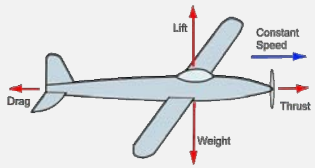
**Learning objectives**

- Understand and apply the concept and equations of equilibrium of a rigid body subject to concurrent and nonconcurrent forces
- Understand and apply the concept of force equipollence

Equilibrium is one of the three fundamental concepts in structural analysis. Under applied load, the structure must be in one of the following equilibrium conditions:

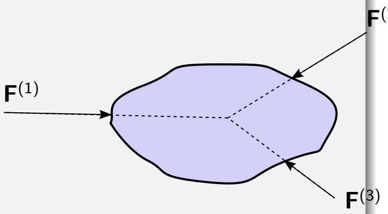
- **Statics:** No net force on the body and the body is at rest (no velocity). Example: building
- **Statics:** No net force and constant velocity (no accelerations). Example: steady-state level flight
- **Dynamics:** Net force is balanced by inertia forces, dynamic equilibrium. Example: maneuvers.

**Example**



# Equilibrium of a body subject to a concurrent force system I:

## Body subjected to concurrent forces




Represent force in cartesian coordinate system using basis vectors $\mathbf{e}_i, i = 1,3$

$$\mathbf{F}^{(k)} = \sum_{i=1}^{3} F_i^{(k)} \mathbf{e}_i$$

In indicial notation (repeated "i" index implies sum sign):

$$\mathbf{F}^{(k)} = F_i^{(k)} \mathbf{e}_i$$

## Static equilibrium (of forces)

To prevent translations in a rigid body, the resultant force on the body must vanish:

$$\boxed{\mathbf{R} = \sum_k \mathbf{F}^{(k)} = \mathbf{0}}$$

In components (three scalar equations):

$$\mathbf{R} = \sum_k \mathbf{F}^{(k)} = \sum_k F_i^{(k)} \mathbf{e}_i = \mathbf{0}$$

$$\boxed{\sum_k F_1^{(k)} = 0}, \boxed{\sum_k F_2^{(k)} = 0}, \boxed{\sum_k F_3^{(k)} = 0}$$

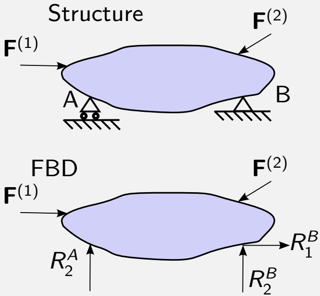
## Implications of force equilibrium

Force equilibrium clearly ensures the absence of linear accelerations (translation equilibrium).

# Free body diagram (FBD):

## FBD

A free body diagram is an idealization of the structure of interest obtained by identifying the external loads and supports and replacing the supports with their corresponding reaction forces. E.g.




## Comments on FBD

- essential step to enforce equilibrium and determine reactions
- especially useful when structure is a collection of interconnected structural components (e.g. truss)

## Procedure

- draw FBD of complete structure
- apply equilibrium equations to determine external reaction forces (when possible)
- draw FBD of substructures or members to determine internal forces
- will use this throughout the year

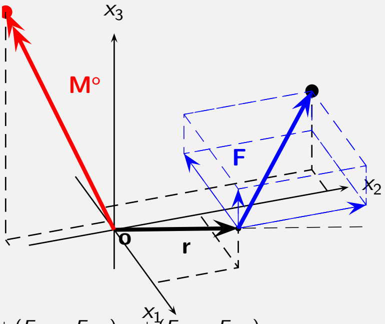
# Equilibrium of a body subject to non-concurrent force system:

We need to devise a way to characterize (and impede) rotations in addition to translations. A proper measure of force capable of producing rotations is the:

## Moment of a force

Intuitively, the moment of a force with respect to a point is the value of the force times the distance from the point of application to the point $F \cdot d$. More precisely in 3D:

$$\boxed{\mathbf{M}^{o} = \mathbf{r} \times \mathbf{F}}, \ [\mathbf{M}] = \mathrm{F} \cdot \mathrm{L}$$



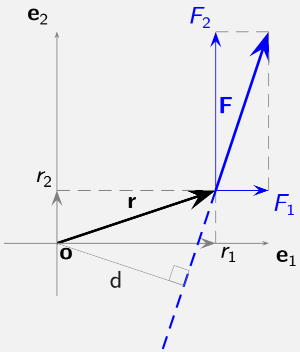

In cartesian components:

$$\mathbf{M}^{o} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ r_1 & r_2 & r_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = \underbrace{(F_3 r_2 - F_2 r_3)}_{M_1^{o}}\mathbf{e}_1 + \underbrace{(F_1 r_3 - F_3 r_1)}_{M_2^{o}}\mathbf{e}_2 + \underbrace{(F_2 r_1 - F_1 r_2)}_{M_3^{o}}\mathbf{e}_3$$

Magnitude of moment: that of the cross product $M^{o} = rF\sin\theta = F\underbrace{r\sin\theta}_{d} = Fd$

# Equilibrium of a body subject to non-concurrent force system:

Special case: 2D

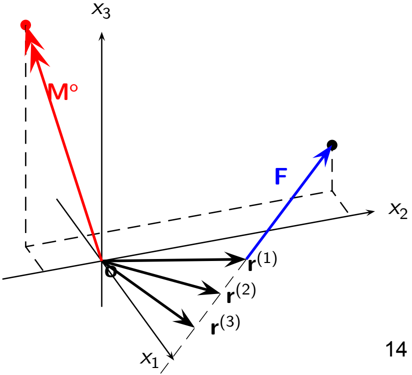



In 2D: $\mathbf{M}^o = M_3\mathbf{e}_3 = (F_2 r_1 - F_1 r_2)\mathbf{e}_3$

# Equilibrium of a body subject to non-concurrent force system:

Principle of Transmissibility: Cross product avoids need to find and use perpendicular distance from force to point. In other words, we can use any position vector **r** measure from point **o** to any point in line of action of the force **F**

$$\mathbf{M}^{\circ} = \mathbf{r}^{(1)} \times \mathbf{F} = \mathbf{r}^{(2)} \times \mathbf{F} = \mathbf{r}^{(3)} \times \mathbf{F}$$




**Principle of transmissibility of force**

The force can be applied at any point along its line of action, it will produce the same moment about point o.

**Principle of moments**

The moment of a force about a point is equal to the sum of the moments of the components of the force about the point

# Equilibrium of a body subject to non-concurrent force system:

## Equilibrium of moments

To prevent rotation of a rigid body, the resultant moment of all the external forces with respect to an arbitrary point must vanish:

$$\boxed{\sum_k \mathbf{r}^{(k)} \times \mathbf{F}^{(k)} = \mathbf{0}}$$

## Summary:

Force equilibrium:

$$\sum_k \mathbf{F}^{(k)} = \mathbf{0}$$

$$\sum_k F_1^{(k)} = 0$$

$$\sum_k F_2^{(k)} = 0$$

$$\sum_k F_3^{(k)} = 0$$

Moment equilibrium:

$$\sum_k \mathbf{r}^{(k)} \times \mathbf{F}^{(k)} = \mathbf{0}$$

$$\sum_k M_1^k = \sum_k (F_3 r_2 - F_2 r_3)^{(k)} = 0$$

$$\sum_k M_2^k = \sum_k (F_1 r_3 - F_3 r_1)^{(k)} = 0$$

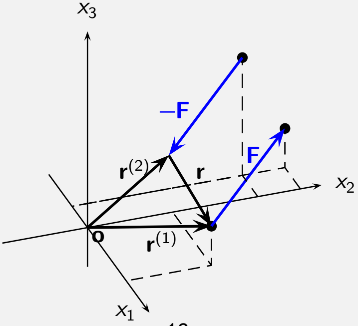
$$\sum_k M_3^k = \sum_k (F_2 r_1 - F_1 r_2)^{(k)} = 0$$

# Equilibrium of a body subject to non-concurrent force system:

## Moment of a couple

A couple is defined as two parallel forces of the same magnitude but opposite directions.
The moment produced by the couple is given by:

$$\mathbf{M} = \mathbf{r}^{(1)} \times \mathbf{F} + \mathbf{r}^{(2)} \times (-\mathbf{F}) = (\mathbf{r}^{(1)} - \mathbf{r}^{(2)}) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$



# Equilibrium of a body subject to non-concurrent force system:

Equipollent force systems

**Definition**

Two force systems are said to be equipollent if they produce the same resultant force and moment (same global STATIC effect) but will produce different deformations and internal forces. Global because local internal forces at different points of the structure would be different even if the two force systems are equipollent, Static (meaning force) because the deformations produced by two equipollent force systems are in general different.

- their static action at a point is equivalent, i.e. they produce the same resultant force and moment
- useful for analysis
- produce same reactions in statics
- same rigid body motion as in dynamics
- BUT, will produce different deformations (e.g. compare a distributed load on a beam vs. a concentrated load in the middle)