1 Problem 1

Let $f_n(t) = \left(\frac{t}{T}\right)^n$ for $t \in [0, T]$. Then we have that $f_n \in C[0, T]$ for $n = 1, 2, ...$ Let $K = \{f_n(t), n = 1, 2, ...\}$. In order to prove K is closed, it suffices to prove $C[0, T] \backslash K$ is open. For any $f \in C[0, T] \backslash K$, assume $\inf_n \|f - f_n\| = 0$. Since $\liminf_n \|f - f_n\| \le \inf_n \|f - f_n\|$, then we have $\liminf_n \|f - f_n\| = 0$. Then there exists a subsequence $\{f_{n_i}, i = 1, 2, ...\}$ such that $\|f - f_{n_i}\| \to 0$ as $i \to \infty$. Thus, we have

$$f(t) = \begin{cases} 0, & \text{if } 0 \le t < T \\ 1, & \text{if } t = T \end{cases}$$

However, f is not continuous and thus $f \notin C[0, T] \backslash K$. Therefore, we have $\inf_n \|f - f_n\| > 0$. There exists a $\epsilon > 0$ such that $B(f, \epsilon) \in C[0, T] \backslash K$, namely, $C[0, T] \backslash K$ is open. Also, we have that

$$\|f_n\| \le 1, \; n = 1, 2, ...$$

Thus, K is also bounded. For two consecutive $f_n(t)$ and $f_{n+1}(t)$, we have

$$\begin{aligned} \|f_n(t) - f_{n+1}(t)\| &= \sup_{0 \le t \le T} \left| (\frac{t}{T})^n (1 - \frac{t}{T}) \right| \\ &= \left(\frac{n}{n+1}\right)^n \frac{1}{1+n} \\ &> \left(\frac{1}{3}\right)^n \frac{1}{1+n} \quad \text{for } n = 1, 2, ... \end{aligned}$$

Then

$$\cup_{n=1}^{\infty} \left\{ B_o \left(f_n(t), \left(\frac{1}{3}\right)^n \frac{1}{1+n} \right) \right\}$$

is an open cover of K, but in this case we can not find a finite subset

$$\cup_{i=1}^{m} \left\{ B_o \left(f_{n_i}(t), \left(\frac{1}{3}\right)^{n_i} \frac{1}{1+n_i} \right) \right\} \quad \text{where } m \text{ is finite.}$$

such that it is a finite subcover of K. Thus, K is not compact.

2 Problem 2

Proof. ($\Rightarrow$) Suppose $f : S_1 \to S_2$ is continuous. For any open set $O \subset S_2$, we have $f^{-1}(O) \subset S_1$. For a fixed $x \in f^{-1}(O)$, we have that $f(x) \in O$. Since O is open, there exists an $\epsilon > 0$ such that $B_o(f(x), \epsilon) \subset O$. Since f is continuous, there exists a $\delta > 0$ such that $f(B_o(x, \delta)) \subset B_o(f(x), \epsilon) \subset O$. Thus, we have $B_o(x, \delta) \subset f^{-1}(O)$. That is, $f^{-1}(O)$ is open.
($\Leftarrow$) Suppose $f^{-1}(O)$ is open in S_1 for every open set $O \in S_2$. For an $x \in S_1$, there is an $\epsilon > 0$ such that $B_o(f(x), \epsilon)$ is an open set in S_2. Thus, we have that $f^{-1}(B_o(f(x), \epsilon))$ is an open set in S_1. For any $x \in f^{-1}(B_o(f(x), \epsilon))$, there exists an $\delta > 0$ such that $B_o(x, \delta) \subset f^{-1}(B_o(f(x), \epsilon))$ which yields $f(B_o(x, \delta)) \subset B_o(f(x), \epsilon)$. Thus, f is continuous. □

3 Problem 3

Proof. Suppose $f_1, f_2, ...$ is a cauchy sequence in $C[0, T]$ with the uniform metric $\|x - y\|$. For an $\epsilon > 0$, there exists an $N > 0$ such that for any $n_1, n_2 > N$, we have

$$\begin{aligned}\epsilon > \|f_{n_1} - f_{n_2}\| &= \sup_{t\in[0,T]} |f_{n_1}(t) - f_{n_2}(t)| \\ &\geq |f_{n_1}(t) - f_{n_2}(t)|, \text{ for any } t \in [0, T].\end{aligned}$$

Thus, for a fixed $t \in [0, T]$, $f_1(t), f_2(t), ...$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we have $f_n(t) \to f(t)$ as $n \to \infty$.

Next, we need to show that $f(t)$ is continuous on $[0, T]$. For $t_1 \in [0, T]$ and any $\delta > 0$, there exists an m large enough, and an $\eta > 0$ such that for every t_2 satisfying $|t_1 - t_2| < \eta$, we have that

$$\begin{aligned}&|f(t_1) - f_m(t_1)| < \frac{\delta}{3} \text{ (by convergence of } \{f_n(t)\} \text{ for a fixed } t.) \\ &|f(t_2) - f_m(t_2)| < \frac{\delta}{3} \text{ (by convergence of } \{f_n(t)\} \text{ for a fixed } t.) \\ &|f_m(t_1) - f_m(t_2)| < \frac{\delta}{3} \text{ (by the continuity of } f_m(t).)\end{aligned}$$

By the triangle inequality, we have

$$|f(t_1) - f(t_2)| \leq |f(t_1) - f_m(t_1)| + |f_m(t_1) - f_m(t_2)| + |f(t_2) - f_m(t_2)| < \delta$$

$f(t)$ is continuous on $[0, T]$, which completes the proof. □

4 Problem 4

Proof. **Part a.**
$M(0) = E[e^0] = 1$. If $M(\theta) < \infty$ for some $\theta > 0$, then for any $\theta' \in (0, \theta]$, we have

$$\begin{aligned} M(\theta') = \mathbb{E}(e^{\theta' X}) &= \int_{-\infty}^{\infty} e^{\theta' x} dP(x) \\ &\leq \int_{0}^{\infty} e^{\theta' x} dP(x) + 1 \\ &\leq M(\theta) + 1 < \infty \end{aligned}$$

Likewise if $M(\theta) < \infty$ for some $\theta < 0$, then for any $\theta' \in [\theta, 0)$, we have

$$\begin{aligned} M(\theta') = \mathbb{E}(e^{\theta' X}) &= \int_{-\infty}^{\infty} e^{\theta' x} dP(x) \\ &\leq \int_{-\infty}^{0} e^{\theta' x} dP(x) + 1 \\ &\leq M(\theta) + 1 < \infty \end{aligned}$$

Part b.
Suppose X has Cauchy distribution, i.e. its density function is

$$f_X(x) = \frac{1}{\pi(1 + x^2)}$$

However, for any $\theta \neq 0$, $\lim_{|x| \to +\infty} \frac{\exp(|\theta x|)}{x^2} = +\infty$, and the function $\frac{\exp(\theta x)}{1+x^2}$ is therefore not integrable.
Part c.
Let X be a random variable with the following probability density function.

$$f_X(x) = \begin{cases} A\exp(x - \sqrt{-x}), & \text{if } x \leq 0 \\ A\exp(-x - \sqrt{x}), & \text{if } x > 0 \end{cases}$$

where $A \doteq 1.10045$ is a normalizing constant. For $\theta \in [-1, 1]$. It is readily verified that

$$M(\theta) = \mathbb{E}[\exp(\theta X)] = A \int_{-\infty}^{0} \exp((1+\theta)x - \sqrt{-x}) dx + A \int_{0}^{\infty} \exp((\theta - 1)x - \sqrt{x}) dx$$

is finite while for any θ outside of $[-1, 1]$, $M(\theta)$ is not finite.
Part d.

Consider a Bernoulli random variable $X \sim Be(1/2)$, so $\mathbb{E}(X) = 1/2$. X satisfies all requirements with $x_0 = 1$. We compute that $M(\theta) = \frac{1}{2}(1 + e^{\theta})$ which is finite for all θ. The rate function is

$$I(x) = \sup_{\theta}\{x\theta - \log(1 + \exp(\theta)) + \log 2\}$$

We differentiate the expression above and obtain

$$\frac{d}{d\theta}(\theta - \log(1 + \exp(\theta) + \log 2)) = x - \frac{\exp(\theta)}{1 + \exp(\theta)}$$

Solving for θ, we obtain $\exp(\theta) = \frac{x}{1-x}$. For $0 \leq x < 1$, the equation admits solution $\theta = \log(\frac{x}{1-x})$ and the rate function is $I(x) = x\log(x) + (1-x)\log(1-x) + \log(2)$. Let $x = 1$. Then $\theta - \log(1 + \exp(\theta)) \leq \theta - \log(\exp(\theta)) \leq 0$, so that $\{\theta - \log(1 + \exp(\theta)) + \log 2\}$ is bounded by $\log 2$ and admits a finite supremum, and $I(1)$ is finite. For $x > 1$ (and $\theta > 0$),

$$\theta x - \log(\frac{1 + \exp(\theta)}{2}) \geq \theta x - \log(\exp(\theta)) \geq (x - 1)\theta$$

Taking $\theta \to +\infty$, we obtain $I(x) = +\infty$. $\square$

5 Problem 5

Proof. Consider two strictly positive sequences $x_n > 0$ and $y_n > 0$. Since $\limsup_n \frac{\log x_n}{n} \leq I$ and $\limsup_n \frac{\log y_n}{n} \leq I$, then for any $\epsilon_1 > 0$, there exists an N such that for any $n > N$, we have

$$\sup_{m \geq n} \frac{\log x_m}{m} \leq I + \epsilon_1, \sup_{m \geq n} \frac{\log y_m}{m} \leq I + \epsilon_1$$

which yields that

$$\max\{\sup_{m \geq n} \frac{\log x_m}{m}, \sup_{m \geq n} \frac{\log y_m}{m}\} \leq I + \epsilon_1$$

Thus, for any $m \geq n$,

$$\max\{\frac{\log x_m}{m}, \frac{\log y_m}{m}\} \leq I + \epsilon_1$$

Taking sup for both sides of the last inequality gives

$$\sup_{m \geq n}\{\max\{\frac{\log x_m}{m}, \frac{\log y_m}{m}\}\} \leq I + \epsilon_1$$

For any $\epsilon_2 > 0$, we can choose n large enough such that

$$\frac{\log 2}{n} + \sup_{m \geq n}\{\max\{\frac{\log x_m}{m}, \frac{\log y_m}{m}\}\} \leq I + \epsilon_1 + \epsilon_2$$

which gives that

$$\sup_{m \geq n}\{\max\{\frac{\log(2x_m)}{m}, \frac{\log(2y_m)}{m}\}\} \leq \frac{\log 2}{n} + \sup_{m \geq n}\{\max\{\frac{\log x_m}{m}, \frac{\log y_m}{m}\}\} \leq I + \epsilon_1 + \epsilon_2$$

Since as $n \to \infty$, we can make ϵ and ϵ_2 both approach to 0. Thus,

$$\lim_{n\to\infty} \sup_{m \geq n} \frac{\log(x_m + y_m)}{m} \leq I$$

that is, $\limsup_{n\to\infty} \frac{\log(x_n + y_n)}{n} \leq I$. □

6 Problem 6

Proof. For any x_1, x_2 and $\alpha \in [0, 1]$, let $x = \alpha x_1 + (1 - \alpha)x_2$, and observe

$$\begin{aligned} I(x) = I(\alpha x_1 + (1-\alpha)x_2) &= \sup_\theta(\theta(\alpha x_1 + (1-\alpha)x_2) - \log M(\theta)) \\ &= \sup_\theta(\alpha(\theta x_1 - \log M(\theta)) + (1-\alpha)(\theta x_2 - \log M(\theta))) \\ &\leq \alpha \sup_\theta(\theta x_1 - \log M(\theta)) + (1-\alpha)\sup_\theta(\theta x_2 - \log M(\theta)) \\ &\leq \alpha I(x_1) + (1-\alpha)I(x_2) \qquad (1) \end{aligned}$$

If $I(x)$ is not strictly convex, there exists $x_1 \neq x_2$, and $\alpha \in (0, 1)$ such that

$$\begin{aligned} &\sup_\theta\{\alpha(x_1\theta - \log M(\theta)) + (1-\alpha)(x_2\theta - \log M(\theta))\} \\ &= \alpha \sup_\theta\{x_1\theta - \log M(\theta)\} + (1-\alpha)\sup_\theta\{x_2\theta - \log M(\theta)\} \end{aligned}$$

However, we know that for every $x \in \mathbb{R}$ there exists $\theta_0 \in \mathbb{R}$ such that $I(x) = \theta_0 x - \log M(\theta_0)$. Moreover, θ_0 satisfies

$$x = \frac{\dot{M}(\theta_0)}{M(\theta_0)}$$

Let $\theta_0 \in \mathbb{R}$ such that $\frac{\dot{M}(\theta_0)}{M(\theta_0)} = \alpha x_1 + (1-\alpha)x_2$. We have

$$\begin{aligned} I(x) &= \theta_0(\alpha x_1 + (1-\alpha)x_2) - \log M(\theta_0) \\ &= \alpha(\theta_0 x_1 - \log M(\theta_0)) + (1-\alpha)(\theta_0 x_2 - \log M(\theta_0)) \end{aligned}$$

Clearly, if either $\theta_0 x_1 - \log M(\theta_0) < \sup_\theta(\theta x_1 - \log M(\theta))$ or $\theta_0 x_2 - \log M(\theta_0) < \sup_\theta(\theta x_2 - \log M(\theta))$, then the equality does not hold. Therefore θ_0 also achieves the maximum for both x_1 and x_2. By first order conditions, we also obtain

$$\frac{\dot{M}(\theta_0)}{M(\theta_0)} = x_1, \ \frac{\dot{M}(\theta_0)}{M(\theta_0)} = x_2$$

which implies that $x_1 = x_2$ and thus gives a contradiction.

□

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