Problem 1 (Adoption Curves)**.** Recall that the equation for the adoption curve in the Bass model is

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}}$$

where p is the innovation rate and q is the imitation rate.

1. Verify the claim made in lecture that $F(t)$ is concave if $p > q$ and is S-shaped (i.e., convex for sufficiently low t and concave for sufficiently high t) if $p < q$.

2. At what time t is the adoption rate $F'(t)$ the highest? Call this time $t^*(p, q)$.

3. Show that $t^*(p, q)$ is always decreasing in p, but can be increasing or decreasing in q depending on parameters. Explain what are the two opposing forces that lead to the ambiguous dependence of $t^*(p, q)$ on q.

Problem 2 (Containment in the SIR Model)**.** Consider the SIR model with basic reproduction number R_0. Suppose that at the beginning of the epidemic, the government vaccinates fraction π of the population, which ensures that they never get sick. (That is, vaccinating an individual immediately moves her to the "removed" state.)

1. Write down the dynamic equations and initial conditions for the SIR model as a function of R_0, π, and an initial infected fraction ι.

2. What fraction of the population ever gets sick in the course of the epidemic?

3. Consider the model with $\gamma = 1$, $\pi = 0.5$, and ι arbitrarily small (so you can treat it as zero in the answer to 2). Suppose that a government in this world is choosing a policy to minimize the number of indviduals who get sick over the course of the epidemic. Due to limited funds, the government must choose between two options. The first is increasing testing, which permanently reduces the value of R_0 by 20%. The second is increasing vaccination, which increases the value of π to 0.75. Show numerically for what values of R_0 the government should invest in testing rather than vaccination.