Note: We are including extra bonus questions to let students work on types of prob-lems that interest them more. There is no expectation that you do all the bonus problems.

Problem 1 (Giant Component)**.** Let $p(n) = \lambda/n$ for all n: that is, expected degree is held fixed at λ.

(a) Suppose that as $n \to \infty$ there is a giant component that fills exactly half the network. What is λ?

(b) For the same random graph, what is the probability that a node has degree exactly 5?

(c) Calculate the fraction of nodes in the giant component that have degree exactly 5. [Hint: for any node i, by Bayes' rule, this equals

$$\frac{\Pr(d_i = 5)\Pr(i \text{ in giant component}|d_i = 5)}{\Pr(i \text{ in giant component})}.$$

You should be able to compute all of these terms.]

(d) Give an intuitive explanation for the difference between the answers to parts (b) and (c).

Problem 2 (Configuration Model)**.** Consider the configuration model with degree distribution $P(d) = 2^{-(d+1)}$ for all $d \geq 0$.

(a) Show that the degree distribution is correctly normalized, meaning that $\sum_{d=0}^{\infty} P(d) = 1$.

(b) What is the average degree of a node?

(c) What is the average number of distance-2 neighbors of a node?

(d) Does the network have a giant component? Why or why not?

Problem 3 (Small World Model)**.** Consider a ring network with n nodes in which each node is connected to its neighbors k steps or less away. There are two popular variants of the "small world" model:

Edge-adding For each pair of nodes that are not linked in this network, add a new edge between them with probability p/n, independently across pairs.

Edge-rewiring For each edge (i, j), with independent probability p, replace this edge with an edge chosen uniformly at random from the set of edges not present in the graph.

(a) Find the degree distribution of the edge adding model. (It suffices to find the asymptotic degree distribution for a given node.)

(b) Show that when $p = 0$, the overall clustering coefficient in both models is given by

$$\mathrm{Cl}(g) = \frac{3k-3}{4k-2}.$$

(c) (*Bonus*-3 points) Show that when $p > 0$, the overall clustering coefficient in the *edge rewiring* model satisfies

$$\frac{3k-3}{4k}(1-p)^3 \leq \mathrm{Cl}(g) \leq \frac{3k-3}{4k-2}(1-p)^3.$$

(d) (*Bonus*-3 points) Write a program to generate small world networks according to the edge adding model with $n = 100$, $k = 5$, and $p = 0.1$. Compute the realized overall clustering coefficient and see if it obeys the bounds for the edge rewiring model from part (c).

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