**Problem 1** (Long-Run Consensus)**.** Consider the DeGroot learning model with $N$ agents with initial belief vector $x(0) = (x_1(0), \ldots, x_N(0))$ and an $N \times N$, non-negative, row stochastic matrix $T$ such that, for every period $t$, we have

$$x(t) = Tx(t-1).$$

(a) Suppose that $N = 3$ and

$$T = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

What properties of this matrix guarantee that, for any initial belief vector $x(0)$, the limit belief $x^* = \lim_{t\to\infty} x(t)$ is well-defined? Compute $x^*$ as a function of $x(0)$.

(b) Suppose that $N = 6$ and

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 \\ \frac{3}{11} & \frac{4}{11} & \frac{4}{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{4}{13} & \frac{3}{13} & \frac{6}{13} \\ 0 & 0 & 0 & \frac{4}{7} & 0 & \frac{3}{7} \end{pmatrix}.$$

Without doing any computations, does $x^* = \lim_{t\to\infty} x(t)$ exist? Why or why not? If so, which components of the vector $x^*$ will be identical, and which components may differ?

(c) Prove that, for any $N$, if there exists an agent $i$ such that $T_{ii} = 1$ and $T_{ji} > 0$ for all $j \neq i$, then $x_j^* \equiv \lim_{t\to\infty} x_j(t)$ is well-defined and equal to $x_i(0)$ for all $j \neq i$.

*[Hint: Let* $\Delta(t) = \max_{j\in N} |x_i(t) - x_j(t)|$ *and let* $\underline{T} = \min_{j\neq i} T_{ji}$*. Prove that* $\Delta(t+1) \leq (1-\underline{T})\Delta(t)$ *for all* $t$*. Show that this implies that each* $x_j(t)$ *must converge to* $x_i(0)$ *as* $t \to \infty$*.]*

**Problem 2** (Clustering)**.** Consider the Erdös-Renyi model with $n > 1$ nodes and link probability $p(n)$, which changes as we add nodes. Let $p(n) = \lambda/n$ for all $n$. Observe that expected degree is held fixed at $\lambda$.

(a) Show that as $n \to \infty$ the expected number of triangles in the network converges to $\frac{1}{6}\lambda^3$. (Recall that a *triangle* is a triple of nodes $(i, j, k)$ such that $g_{ij} = g_{ik} = g_{jk} = 1$.) Thus, the expected number of triangles hardly depends on $n$ (once $n$ is large). Explain how this is possible.

(b) Show that for large $n$ the expected number of connected triples in the network is approximately $\frac{1}{2}n\lambda^2$. (Recall that a *connected triple* is a triple of nodes $(i, j, k)$ such that $g_{ij} = g_{ik} = 1$.)

(c) Define the *clustering coefficient* for a random network to be the probability that two neighbors of a node are also neighbors of each other. Compute the clustering coefficient for the Erdös-Renyi model with $p(n) = \lambda/n$.

**Problem 3** (Phase Transition)**.** Consider again the Erdös-Renyi model with $n > 1$ nodes and link probability $p(n)$. Let $A$ denote the event that node 1 has at least $l \in \mathbb{N}$ neighbhors. Show that there is a phase transition for this event with the threshold function $t(n) = \lambda/n$ for some $\lambda > 0$. [Hint: You may need to use the fact that $\left(1 + \frac{x}{n}\right)^n \approx \exp(x)$ when $n$ is large for any $x \in \mathbb{R}$.]

*Bonus*: Using a computer language of your choice, code a program that simulates Erdös-Renyi graphs with $n$ nodes and connection probability $p$. Use a simulation with this program to illustrate the phase transition as $p(n)$ crosses the threshold $t(n) = 1/n$. If you "inspect" the networks produced, what other properties do you notice on either side of the threshold?

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