6.207/14.15/14.150 Spring 2022, Midterm Exam

Instructions

- Each problem has 10 points.
- You have 120 minutes to complete the exam.
- You may not use (and will not need) a calculator.
- **Please print your name on the first page of the document you scan and upload.**

Name ____________________________

Problem 1 Consider an undirected network. Recall that Katz-Bonacich centrality with parameter α is defined by the equation $\mathbf{c} = \alpha\mathbf{g}\mathbf{c} + \mathbf{1}$, where $\mathbf{c}$ is a $n \times 1$ vector and $\mathbf{g}$ is an $n \times n$ adjacency matrix.

(a) (**4 points**) Show that Katz-Bonacich centrality can also be written in the series form $\mathbf{c} = \mathbf{1} + \alpha\mathbf{g}\mathbf{1} + \alpha^2\mathbf{g}^2\mathbf{1} + \ldots$. (You may assume that this series converges.) Give an intuitive explanation of this formula.

(b) (**3 points**) Suppose that two nodes i and j satisfy $\mathbf{c}_i > \mathbf{c}_j$ for all sufficiently small values of the parameter α. What, if anything, can we conclude about the degrees of nodes i and j?

(c) (**3 points**) Draw a network and label two nodes i and j for which you would guess that $\mathbf{c}_i > \mathbf{c}_j$ for sufficiently small values of α but $\mathbf{c}_i < \mathbf{c}_j$ for sufficiently large values of α. Explain your guess. You do not need to do any calculations or verify that your guess is correct.

Problem 2 Consider the DeGroot learning model with N agents with initial belief vector $x(0) = (x_1(0), \ldots, x_N(0))$ and an $N \times N$, non-negative, row-stochastic matrix T such that, for every period t, we have

$$x(t) = Tx(t-1).$$

(a) (**5 points**) Suppose that $N = 3$ and

$$T = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Compute the limit belief $x^* = \lim_{t\to\infty} x(t)$ as a function of $x(0)$. Which of the three agents is the most "influential"?

(b) (**5 points**) Suppose that the matrix T is strongly connected and aperiodic, and that there is an agent i and a number t such that $\left(T^t\right)_{ji} = 0$ for all j (including $j = i$), where $\left(T^t\right)_{ji}$ denotes the (j, i) component of the t^{th} power of the matrix T. Prove that, for any two vectors of initial beliefs $x\left(0\right)$ and $\hat{x}\left(0\right)$ such that $x_j\left(0\right) = \hat{x}_j\left(0\right)$ for all $j \neq i$, we have $\lim_{t\to\infty} x\left(t\right) = \lim_{t\to\infty} \hat{x}\left(t\right)$. (You may appeal to any results proved in class, provided you cite them correctly.)

Problem 3 Consider the standard SIR model with basic reproduction number $R_0 > 1$. Suppose that the government has the ability to implement a *lockdown*. Assume that a lockdown prevents all new infections until the currently infected share of the population falls to a small number $\varepsilon > 0$, at which point the lockdown is lifted. Assume that the government can only implement a lockdown once (for example, this could be because society will not tolerate multiple lockdowns). The government's problem is to choose the timing of the lockdown so as to minimize the long-run share of the population that ever gets infected, $\lim_{t\to\infty} R(t)$.

(a) (**4 points**) When should the government implement the lockdown? Your answer should take the form of a rule that the the government can implement if at each point in time t it knows what fraction of the population is susceptible, infected, and recovered. Assuming that the government locks down at the optimal time, what is $\lim_{t\to\infty} R(t)$? (Hint: This is a 1-line calculation.)

(b) (**3 points**) Now consider the behavioral SIR model covered in Lecture 11. Suppose the government's objective remains the same. Again, when should the government implement the lockdown? Will this point in time be reached earlier or later than in the standard SIR model? What is $\lim_{t\to\infty} R(t)$?

(c) (**3 points**) How does the long-run share of the population that ever gets infected differ in parts (a) and (b)? Intuitively, would you say that society is better-off in part (a) or part (b)? Explain.