

# Lecture 19: Cooperation in Repeated Games, Communities, and Networks

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## Trust and Cooperation

Last class introduced dynamic games on networks in the context of bargaining and intermediation.

Today, another question about dynamic interactions: how is trust and cooperation sustained in communities and social networks?

Important issue: It's widely believed that trust and social norms of cooperation and reciprocity are critical for supporting good social outcomes in social networks, businesses, and societies.

- ▶ Social networks: information-sharing (e.g. Granovetter on finding jobs), risk-sharing (e.g. in the developing world), managing public goods and environmental resources.
- ▶ Businesses: sharing information and credit (e.g. Munshi on Indian diamond industry), long-run business relationships.
- ▶ Societies: “high trust” vs. “low trust” societies, “social capital,” “informal institutions” in economic development.

## Questions about Trust and Cooperation

- ▶ What strategies can groups use to support trust/cooperation?
- ▶ What properties of a group or its members are conducive to supporting a high level of cooperation?
- ▶ Which members of a group (e.g. which network positions) are most important for supporting cooperation?
- ▶ What kinds of “institutions” (e.g. ways of sharing information, shaping preferences, or punishing misbehavior) can help support cooperation?

## Modeling Trust and Cooperation: Repeated Games

There are different approaches to modeling trust and cooperation in groups. The main one in economics and related fields is **repeated games**: we model a long-run relationship as a set of players playing the same game over and over.

- ▶ Repeated games are a natural framework for modeling **reciprocity**: you scratch my back today, I'll scratch yours tomorrow.
- ▶ Another important approach to modeling trust and cooperation is using ideas from evolution to think about how **altruism**—an intrinsic taste for helping others, or punishing people who misbehave—might evolve. This is an important topic in evolutionary biology and is potentially complementary with the repeated games approach.

We'll cover the basic model of cooperation of repeated games, and apply it to study cooperation in communities and networks.

## Repeated Games: Model

A **repeated game**  $G^T(\delta)$  is defined by a (finite) static game  $G$  with action sets  $A_1, \dots, A_n$  and payoff functions  $g_i : \prod_{i=1}^n A_i \rightarrow \mathbb{R}$  (called the **stage game**), a **time horizon**  $T$  (which can be finite or infinite), and a discount factor  $\delta \in [0, 1]$ .

In every period  $t = 1, 2, \dots, T$ , the player simultaneously choose actions  $(a_1^t, \dots, a_n^t)$ , after observing all previous actions.

Payoffs are

$$u_i = \sum_{t=1}^T \delta^{t-1} g_i(a_1^t, \dots, a_n^t) \quad \text{for all } i \in N.$$

**Note:** Players maximize present discounted value.

- ▶ If  $T = \infty$ , need  $\delta < 1$  to keep payoffs from blowing up.
- ▶ If  $T$  is finite, payoffs are well-defined even if  $\delta = 1$ .

We look for the subgame perfect equilibria (SPE).

## Example: Repeated Prisoner's Dilemma

Suppose the stage game  $G$  is the prisoner's dilemma (PD):

	$C$	$D$
$C$	1, 1	-1, 2
$D$	2, -1	0, 0

The corresponding repeated game is called the **repeated prisoners' dilemma**: players play the PD over and over again, observing all past actions before choosing actions for the current period.

In the one-shot PD, the unique equilibrium is  $(D, D)$ .

Our first question is, can repetition lead to cooperation in the PD, i.e. is there a SPE in the repeated PD where the players ever play  $(C, C)$ ?

## Finitely-Repeated PD

First ask this question in the context of the **finitely-repeated** PD: the time horizon  $T$  is a finite number.

Recall: we find the SPE of finite-horizon dynamic games by backward induction.

Let's apply backward induction to the finitely-repeated PD.

## Finitely-Repeated PD (cntd.)

In the last period,  $D$  is a dominant strategy regardless of prior play.

- ▶ So, in every subgame starting in period  $T$ , the unique NE is  $(D, D)$ .

Now move to period  $T - 1$ .

- ▶ By backward induction, we know that, no matter what is played in period  $T - 1$ ,  $(D, D)$  will be played in period  $T$ .
- ▶ Therefore,  $D$  is also a dominant strategy in period  $T - 1$  (again regardless of prior play).
- ▶ So, in every subgame starting in period  $T - 1$ , the unique SPE is for each player to play  $D$  in period  $T - 1$ , and then  $D$  again in period  $T$  (no matter what her opponent plays in period  $T - 1$ ).

By induction, the unique SPE is for each player to play  $D$  in every period, regardless of prior play.

- ▶ **Conclusion:** any finite repetition of the PD, no matter how long, does not lead to cooperation!

## A More General Result

The same reasoning as for the finitely-repeated PD applies to any repeated game where the stage game has a unique NE.

### Theorem

*If a static game  $G$  has a unique NE then, for any  $T < \infty$  and  $\delta \in [0, 1]$ , the  $T$ -times repeated game  $G^T(\delta)$  has a unique SPE. In the SPE, the unique NE of  $G$  is played in every period.*

**Proof:** By backward induction, similar to the PD.

## Infinitely-Repeated PD

Now consider the repeated PD with  $T = \infty$ .

Note: one interpretation of the discount factor  $\delta$  is that the time horizon is actually **uncertain**, and  $\delta$  represents the probability that the game lasts another period.

The key difference between the infinitely-repeated PD and the finitely-repeated PD is not that the infinitely-repeated game literally lasts forever, but rather that there is no commonly known endpoint of the game (so the outcome is not determined by backward induction).

This will make a big difference.

## Trigger Strategies

A simple kind of strategies in the infinitely-repeated PD are **trigger strategies**.

- ▶ Each player starts off taking  $C$ .
- ▶ Each player keeps taking  $C$  so long as both players have always played  $C$ .
- ▶ If anyone ever plays  $D$ , both players switch to playing  $D$  forever.

If both players play grim trigger, what is the outcome of the game?

# Trigger Strategies in the Repeated PD

## Theorem

*In the infinitely repeated PD, grim trigger strategies are a SPE iff  $\delta \geq \frac{1}{2}$ .*

So, the **infinite** repetition of the PD can support cooperation, so long as the players are sufficiently patient.

## Proof

By the “one-shot deviation principle,” to check that a strategy profile is a SPE in a repeated game, we have to check that no player can gain by deviating from it in any single period (for any history of past play).

If your opponent plays grim trigger, her strategy depends only on whether everyone has always played  $C$  so far, or whether someone has played  $D$ .

So there are only two kinds of histories to check: those where everyone has always played  $C$ , and those where someone has played  $D$ .

## Proof (cntd.)

If someone has already played  $D$ , your opponent will play  $D$  forever no matter what you do.

Against this strategy of your opponent, your best response is to play  $D$  in every period.

This is exactly what grim trigger prescribes.

So, regardless of  $\delta$ , a player cannot gain by deviating at a history where someone has already played  $D$ .

## Proof (cntd.)

Suppose instead everyone has always played  $C$  so far.

- ▶ Since your opponent plays grim trigger, she will play  $C$  today and follow grim trigger in the future.

If you play  $C$ :

- ▶ You get payoff 1 today.
- ▶ Following grim trigger, you will play  $C$  in every future period, and hence so will your opponent.
- ▶ So you will also get payoff 1 in every future period.

If you play  $D$ :

- ▶ You get payoff 2 today.
- ▶ Following grim trigger, you will play  $D$  in every future period, and so will your opponent.
- ▶ So you will get payoff 0 in every future period.

Grim trigger prescribes  $C$ . So you gain from deviating iff you prefer (2 today, 0 in every future period) to (1 today, 1 in every future period).

## Proof (cntd.)

Payoff from (2 today, 0 in every future period):

$$2 + \delta (0) + \delta^2 (0) + \dots = 2.$$

Payoff from (1 today, 1 in every future period):

$$1 + \delta (1) + \delta^2 (1) + \dots = \frac{1}{1 - \delta}.$$

So you do not gain from deviating (and thus grim trigger is a SPE)  
iff

$$\begin{aligned} 2 &\leq \frac{1}{1 - \delta} \\ &\iff \\ \delta &\geq \frac{1}{2}. \end{aligned}$$

## Multiplicity of Equilibria

Can we conclude from this that, if players are sufficiently patient, they **will** cooperate in the infinitely-repeated PD?

Not necessarily: cooperation is **a** SPE, but there are many others.

What are some?

- ▶ Always  $D$  (for any  $\delta$ )
- ▶ Always  $D$  for the first 17 periods, then grim trigger (for  $\delta \geq \frac{1}{2}$ )
- ▶ Always  $D$  in even periods, “grim trigger restricted to the odd period” in odd periods (for  $\delta \geq \frac{1}{\sqrt{2}}$ ).
- ▶ Alternating  $(C, D)$  and  $(D, C)$ , enforced by switching to Always  $D$  after any deviation (for  $\delta \geq \frac{1}{2}$ ).

In some games repetition can even lead to **worse** outcomes than any stage-game NE.

Nonetheless, grim trigger strategies show that players **can** cooperate via reciprocity in infinitely repeated games, even if we can't unambiguously predict that they **will**.

## The Folk Theorem

More generally, a set of results known as the “folk theorem for repeated games” says that, if players are patient enough, **almost any** possible payoffs can arise in an SPE of a repeated game.

The simplest version of this result is:

### Theorem

*Consider any action profile  $\mathbf{a}$  such that there exists a stage-game NE  $\mathbf{a}^{NE}$  where  $u_i(\mathbf{a}^{NE}) < u_i(\mathbf{a})$  for all  $i \in I$ . There exists some  $\bar{\delta} < 1$  such that, for all  $\delta > \bar{\delta}$ , there is a SPE where  $\mathbf{a}$  is played in every period.*

### Proof sketch:

- ▶ Use trigger strategies that start with everyone playing  $\mathbf{a}$  but switch to everyone playing  $\mathbf{a}^{NE}$  if there is any deviation.
- ▶ For sufficiently high  $\delta$ , can't gain by deviating before the switch, because  $u_i(\mathbf{a}^{NE}) < u_i(\mathbf{a})$  for all  $i$ .
- ▶ Regardless of  $\delta$ , can't gain by deviating after the switch, because if everyone else is playing a stage-game NE forever, you should play it, too.

## Social Norms and Decentralized Repeated Games

Cooperative equilibria in repeated games can serve as a model of beneficial social norms.

- ▶ We define cooperative behavior as incurring a cost to benefit someone else.
- ▶ Most sociologists view social norms as “internalized” to the point where they are followed even by people who expect no future benefit from following the norm. If so, this is outside our definition of “cooperation,” and is simply optimal behavior given one’s “socialized” preferences.
- ▶ We instead model social norms as equilibria in repeated games: people follow the norm because they expect future benefits from doing so.
- ▶ However, we usually view social norms as arising in **decentralized** repeated games, i.e. games where not everyone in society interacts with each<sup>19</sup> other in every period.

## Repeated Games with Random Matching

A simple model of a decentralized society is a **repeated game with random matching**.

- ▶ There is a population of  $N$  players.
- ▶ Each period  $t = 1, 2, \dots$ , the players randomly split up into pairs to play a symmetric, two-player stage game (e.g. the PD).
- ▶ Let  $j(i, t)$  denote player  $i$ 's partner in period  $t$ .
- ▶ Player  $i$ 's period- $t$  payoff is  $u(a_i^t, a_{j(i,t)}^t)$ .
- ▶ Player  $i$ 's total per-period payoff is

$$\sum_{t=1}^{\infty} \delta^{t-1} u(a_i^t, a_{j(i,t)}^t).$$

- ▶ Key new assumption: players are anonymous and only observe actions in their own matches. (A form of limited information.)

## Limited Information

Is cooperation possible in repeated games with random matching and limited information?

One's first thought might be that this is impossible (at least if  $N$  is large), because any one player can defect and then “disappear into the crowd.”

- ▶ Indeed, you're asked to show on the problem set that this is correct if we first fix any  $\delta < 1$  and then take  $N \rightarrow \infty$ .

However, if we fix any finite  $N$  and then take  $\delta$  large, a player who deviates cannot disappear into the crowd completely: she will eventually meet players who met players who... met the player she deviated against. This gives some hope that collective punishment strategies may be effective.

## Contagion Strategies

The anonymous matching version of trigger strategies are called **contagion strategies**.

Each player uses the following strategy:

- ▶ Play  $C$  so long as you and everyone you've ever met have always played  $C$ .
- ▶ If you or anyone you've ever met has ever played  $D$ , play  $D$  forever.

Intuitively, once any player deviates, the “contagion” of playing  $D$  spreads throughout the population, until everyone is playing  $D$ .

# Cooperation with Contagion Strategies

## Theorem

*In the prisoner's dilemma with anonymous random matching, contagion strategies are a NE when players are sufficiently patient.*

## Proof Sketch

- ▶ Suppose the players use contagion strategies.
- ▶ If  $D$  is played for the first time in some period  $t$ , then for all  $\varepsilon > 0$ , there exists  $T(\varepsilon)$  such that, with probability at least  $1 - \varepsilon$ , everyone else starts playing  $D$  by period  $t + T(\varepsilon)$ . (Since  $N$  is finite.)
- ▶ If players are very patient, this implies that deviating and starting contagion is unprofitable: with high probability, deviating gives at best a payoff of 2 instead of 1 for  $T(\varepsilon)$  periods, but then a payoff of 0 instead of 1 thereafter.

## What Have We Learned?

Let's return to our motivating questions and see what we've learned about group cooperation (and then turn to some empirical examples).

What strategies can be used to support cooperation?

- ▶ With full information, can support cooperation by threatening widespread breakdown of cooperation following a deviation. But different strategies can also work, including some that punish only those who deviate themselves.
- ▶ With limited information (i.e. anonymous matching), more difficult to punish only the deviator. Collective punishment (contagion) is more promising.
- ▶ More information makes targeted punishments more feasible.

## What Have We Learned? (cntd.)

What kinds of groups are good at supporting cooperation?

- ▶ Ones where players are patient, interact with each other frequently, or expect the relationship to last for a long time: these all correspond to a high value for  $\delta$ .
- ▶ Ones where high-quality information about group members' past behavior is readily available. This facilitates effective punishments for deviation.

Which members of a group are most important for cooperation?

- ▶ We can't answer this question with the symmetric random matching game we've considered so far.
- ▶ Next, consider repeated games played on networks, where we will be able to answer this question.

What kinds of institutions can help support cooperation?

- ▶ Intuitively, those that in effect increase  $\delta$  or improve information. Let's see some examples.

## The Value of Small and Close-Knit Groups

In social science, there are many examples of smaller or more close-knit groups being able to support higher levels of trust and cooperation than larger and more diffuse groups.

A simple way to model this is to note that the critical discount factor needed for contagion strategies to support cooperation in the repeated PD is increasing in  $N$ .

- ▶ Larger  $N$ 
  - ⇒ takes longer for most other players to start Defecting after you Defect
  - ⇒ a player's payoff after Defecting is higher
  - ⇒ must be more patient to support cooperation.
- ▶ In this sense, cooperation is easier in smaller groups.

## The Value of Small and Close-Knit Groups (cntd.)

Alternatively, suppose we fix  $N$  but assume that in every period each player “gossips” with  $K$  other players by telling them the outcomes of all of her past matches.

- ▶ Modify contagion strategies to specify that you switch to  $D$  if you ever hear about anyone playing  $D$ , in addition to directly meeting someone who plays  $D$ .
- ▶ Then the higher is  $K$ , the fast contagion spreads, and the lower is the critical discount factor for supporting cooperation.
- ▶ Here,  $K$  is a crude measure of how “close-knit” the group is (e.g. perhaps we gossip with each other whenever we interact socially, which is not directly related to our “economic” interactions of playing the PD with different people).
- ▶ In this sense, cooperation is easier in close-knit groups.

When we introduce explicit network structure into the community, we'll be able to develop richer and more subtle versions of these results.

## Application: The Maghribi Traders

In European economic history, the **commercial revolution** refers to a period from the 11th to 14th century where long-distance trade in European and the Mediterranean resumed at large scale for the first time since the fall of the Roman Empire.

## Application: The Maghribi Traders (cntd.)

- ▶ Long-distance trade requires large investments in each voyage or caravan.
- ▶ A merchant who makes such an investment must employ an overseas agent in the destination country to provide services like loading/unloading ships, paying fees/bribes, and marketing the goods.
- ▶ By definition, these agents are far away.
- ▶ By the nature of the services they must provide, they have opportunities to cheat the funding merchants.
- ▶ Legal enforcement at the time was typically minimal, especially for things like enforcing the property rights of overseas investors.
- ▶ So, an important puzzle is: how did long-distance trade get going in such circumstances?

In an important series of papers, economist Avner Greif argues that gossip among merchants and the threat of excluding dishonest agents was the critical factor in enabling long-distance trade.

## The Maghribi Traders (cntd.)

Greif conducted a historical and game-theoretic study of a particular group of medieval traders: the Maghribi, an important group of Jewish traders in the Western Mediterranean.

- ▶ As Jews living in majority-Muslim countries, the Maghribis were a close-knit group, and they interacted and shared information frequently.
- ▶ Greif found evidence that whenever an overseas agent cheated a Maghribi, he was boycotted by the Maghribi as a group (not just the particular trader he cheated).
- ▶ This multilateral punishment strategy gives much stronger incentives for overseas agents to behave than bilateral punishment strategies would.
- ▶ This is what allowed cooperation to arise given the actual limited time-horizon of agents and relatively infrequent interactions between any given pair of agent and merchant (i.e. relatively low  $\delta$ ).

## Similar Examples

The Maghribi traders example is one where a close-knit ethnic or religious community has an advantage in high-stakes trade in the absence of effective enforcement, due to the ability to gossip about and subsequently ostracize deviators.

- ▶ In these situations, “community enforcement” substitutes for “legal enforcement.”

Several other well-known examples of this have been studied by social scientists and legal scholars:

- ▶ Orthodox Jews in the New York diamond industry.
- ▶ Ethnic Chinese traders in South-East Asia.

## Another Example: The Law Merchant

A second example also comes from the Commercial Revolution.

In addition to caravans and sea voyages, the other main setting for long-distance trade in this period was “fairs.”

- ▶ Merchants bring goods to the fair.
- ▶ Typically sell goods not for cash, but for promissory notes to be paid at the next fair.
- ▶ Why does a merchant pay back his notes?
- ▶ In this context, merchants come from all over Europe, so decentralized gossip as in the Maghribi case is not likely to provide enough information.

## The Law Merchant (cntd.)

The historical “solution” that arose in this case was the **law merchant**: private judges who attended the fairs.

Apparently, the law merchant system worked roughly like this:

- ▶ Before a merchant trades his goods for a promissory note, he queries the judge as to whether his partner has any outstanding notes.
- ▶ If yes, don't trade. If no, trade and record the note in the judge's book.
- ▶ When the promissory note is later paid, record that in the book as well.

The judge thus facilitates multilateral punishment without the need for widespread gossip.

- ▶ Each merchant doesn't need to learn everything about everyone. Just needs to ask if the current-period partner has any outstanding notes.
- ▶ See Milgrom, North, and Weingast (1990), *The Role of Institutions in the Revival of Trade*.

# Decentralized Record-Keeping: From the 11th Century to the 21st Century

The law merchant is an example of **decentralized record-keeping**: each trader voluntarily enters his trading information in the ledger, and queries the ledger to learn about his potential trading partners.

A potential problem with the law merchant system:

- ▶ The system is not completely decentralized, because the judge is solely responsible for the accuracy of the ledger.
- ▶ Could a merchant with an unpaid note bribe the judge to erase it from the ledger?
- ▶ If so, the system breaks down. However, perhaps the long-run incentives of the judge himself would keep him honest. (Or the incentives of other judges, who care about keeping each judge honest so that judges in general are trusted.)

## Decentralized Record-Keeping: From the 11th Century to the 21st Century (cntd.)

If no single individual's long-run incentives are powerful enough to keep her trustworthy, the keeping of the ledger must also be decentralized.

- ▶ Creating a trustworthy, completely decentralized ledger is the goal of cryptographic technologies like permissionless blockchains (which underlie Bitcoin).
- ▶ A key difference between blockchain and (quasi-)legal institutions like courts and the law merchant is that in the latter case some individuals are more trusted than others.
- ▶ However, lack of a trusted authority causes its own challenges, namely determining who gets to update the ledger.
- ▶ Bitcoin addresses this problem through the Proof-of-Work protocol, which is costly and energy-intensive.
- ▶ Assessing and designing better protocols for decentralized record-keeping is an active area of research at the intersection of computer science and economics.

# Cooperation and Trust in Networks

So far, we have studied how communities can support cooperation as an equilibrium of a decentralized repeated game.

But we have ignored network structure, which can matter in several ways:

- ▶ We know that network structure affects speed of diffusion of information. If what's diffusing is information about who has deviated, this affects how quickly a deviator can be punished, which affects how much cooperation can be supported.
- ▶ Some individuals are more strongly embedded in the community (central) than others. These individuals can be expected to behave more cooperatively.

## Cooperation on a Network: Framework

Suppose the players are arranged on a network with edges  $E$ .

Each period  $t = 1, 2, \dots$ , each player  $i$  chooses a **level of cooperation** (or **effort**)  $x_i \in \mathbb{R}_+$ .

Assume effort benefits everyone, but effort is only observed by one's neighbors in the networks.

- ▶ Network matters for **information**, not directly for payoffs.
- ▶ Normalize the cost of exerting effort  $x_i$  to simply  $x_i$  itself.
- ▶ Assume effort  $x_i$  benefits everyone else by some amount  $f(x_i)$ , where  $f$  is an increasing and concave function.

Stage game payoffs at effort profile  $x = (x_1, \dots, x_n)$  are given by

$$u_i(x) = \sum_{j \neq i} f(x_j) - x_i.$$

The game is played repeatedly with discount factor  $\delta$ .

## Maximum Cooperation

What is the maximum level of cooperation that can result in any Nash equilibrium?

We will assume the players use contagion strategies: There is some vector of “correct” effort levels  $x = (x_1, \dots, x_n)$  such that

- ▶ Each player  $i$  exerts effort  $x_i$  in every period, so long as every player  $j$  she observes has always exerted effort  $x_j$ .
- ▶ If player  $i$  ever sees any player  $j$  take an effort level  $x_j' \neq x_j$ , she takes effort 0 forever.

**Note:** Allow  $x_i \neq x_j$ , because players can have (endogenously) different incentives to cooperate due to their network position.

## Maximum Cooperation: Idea

Given that players use contagion strategies, what is the component-wise maximum vector of effort levels  $x$  that can be attained in equilibrium?

If player  $i$  deviates, she saves her effort cost of  $x_i$  in every period.

But she also gradually loses the benefits of others' cooperation, as contagion started by her defection spreads throughout the network.

- ▶ If player  $i$  deviates today, then player  $j$  stops cooperating in  $d(i, j)$  periods.

## Maximum Cooperation: Conditions

Putting this together, a vector of cooperation levels  $x = (x_1, \dots, x_n)$  can be supported in equilibrium if and only if, for each player  $i$ , we have

$$\underbrace{\frac{1}{1-\delta} \sum_{j \neq i} f(x_j) - x_i}_{\text{equilibrium payoff}} \geq \underbrace{\sum_{t=0}^{\infty} \delta^t \sum_{j: d(i,j) > t} f(x_j)}_{\text{payoff if stop cooperating}}$$
$$\iff \underbrace{x_i}_{\text{per-period gain if stop cooperating}} \leq \underbrace{\sum_{j \neq i} \delta^{d(i,j)} f(x_j)}_{\text{per-period loss if stop cooperating}} .$$

## Maximum Cooperation: Solution

Therefore, the greatest vector of cooperation levels  $x = (x_1, \dots, x_n)$  that can be supported in equilibrium given by the (component-wise) greatest solution to the system of equations

$$x_i = \sum_{j \neq i} \delta^{d(i,j)} f(x_j) \text{ for all } i.$$

**Note:** At solution,  $x_i$  is different for different players  $i$ , depending on their position in the network.

Aside: Does such a solution always exist?

Yes. Intuitive argument:

- ▶ Fix the cooperation levels of everyone except player  $i$  at a high level, and see how much player  $i$  is willing to cooperate in response to this. Do this for every player  $i$ .
- ▶ Then see how much everyone is willing to cooperate in response to *those* cooperate<sup>42</sup> levels.
- ▶ Iterate until we converge to a fixed point.

# Maximum Cooperation: Interpretation

Formula for maximum cooperation:

$$x_i = \sum_{j \neq i} \delta^{d(i,j)} f(x_j) \quad \text{for all } i$$

We can get several insights from this formula.

When is it possible to support more cooperation?

- ▶ When players are more patient:  $x_i$ 's are increasing in  $\delta$ .
- ▶ When the network is “denser”:  $x_i$ 's are decreasing in  $d(i, j)$  for all  $i, j$ .

## Maximum Cooperation: Interpretation (cntd.)

Formula for maximum cooperation:

$$x_i = \sum_{j \neq i} \delta^{d(i,j)} f(x_j) \quad \text{for all } i$$

Fixing a network, which individuals cooperate more (i.e. for which players  $i$  is  $x_i$  higher)?

- ▶ Having **more** distance- $t$  neighbors (i.e. larger  $N_i(t)$  sets) encourages cooperation by player  $i$ .
- ▶ Having distance- $t$  neighbors **who themselves have more distance- $t$  neighbors** (i.e. large  $x_j$ 's for  $j \in N_i(t)$ ) also encourages cooperation by player  $i$ .
- ▶ A player has strong incentives to cooperate if she is closely connected to other players who themselves have strong incentives to cooperate.
- ▶ Similar to Katz-Bonacich centrality with  $g_{ij} = \delta^{d(i,j)}$ , but a non-linear relationship since  $f$  is non-linear.

## Recursive Centrality

**Recursive centrality** formalizes this concept of one player having “more, and more central” neighbors than another.

- ▶ Unlike standard centrality measures, it only is a partial order.
- ▶ However, we will be able to conclude that, if  $i$  is more central than  $j$ , then  $x_i \geq x_j$  **regardless** of the discount factor  $\delta$  and the shape of the “benefit function”  $f$ .

## Recursive Centrality: Definition

Say  $i$  is **1-more central** than  $j$  if  $i$  has more distance- $t$  neighbors than  $j$  for every  $t$ .

- ▶ That is,  $|N_i(t)| \geq |N_j(t)|$  for all  $t$ .

Recursively, say  $i$  is **s-more central** than  $j$  if for every  $t$  there is an injection  $\psi : N_j(t) \rightarrow N_i(t)$  such that, for each  $k \in N_j(t)$ ,  $\psi(k)$  is  $(s - 1)$ -more central than  $k$ .

- ▶ Intuitively, “ $i$ ’s distance- $t$  neighbors are more central than  $j$ ’s distance- $t$  neighbors.”

Finally, say  $i$  is **more central** than  $j$  if  $i$  is  $s$ -more central than  $j$  for all  $s$ .

# Centrality and Maximum Cooperation

## Theorem

*If  $i$  is more central than  $j$ , then  $x_i \geq x_j$ .*

## Proof idea:

- ▶ Recall how the maximum cooperation vector  $x$  was characterized by iterating a best response-like function starting from a vector of very high cooperation levels.
- ▶ If  $i$  is more central than  $j$ , then  $x_i$  stays above  $x_j$  at every step of this iteration, and hence converges to a higher value.

**Note:** It can happen that  $i$  is 1-more central than  $j$  but  $x_i < x_j$ .

- ▶ Compare the center of an  $n + 1$ -player star with a member of an  $n$ -player clique.

## Cooperation on Networks: Summary

In repeated cooperation games on networks, the maximum level of cooperation is supported by contagion strategies, where each player balances her cost of effort against the benefits from others' effort that are lost if she deviates.

Maximum cooperation is greater when players are more patient and the network is more close-knit (short path lengths).

Players who occupy more central positions in the network (in the sense of having more and more central distance- $t$  neighbors) cooperate more.

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