Lecture 15: Network Effects in Markets and Games

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Network Effects: Examples

Many products are more desirable when more people use them.

Older examples:

- telephone
- fax machine

Contemporary examples:

- operating systems
- search engines
- messaging apps
- social media
- video game platforms
- credit cards

Examples (cntd.)

For some of these, particularly valuable if your friends/acquaintances use them ("local network effects").

- telephone
- fax machine
- messaging apps
- social media
- video games

For others, overall market share matters more than who uses them (classical "network effects").

- operating systems
- search engines
- credit cards

Network Effects

A social or economic environment is said to exhibit **network effects** when the optimal action of an individual depends on some average of the actions of others.

- E.g. a brand of computer or search engine becomes more desirable as it becomes more popular.
- There are **local network effects** if what matters is an average of the actions of one's neighbors only.
- E.g. wanting to use the same messenging app as one's friends.

Thus, **confusingly**, the term "network effects" often doesn't have much to do with mathematical networks per se: when networks show up explicitly, we usually have "local network effects."

- Nonetheless, the types of social interactions modeled as network effects fit naturally into this course, so we will study both "network effects" and "local network effects."

Network Effects: Plan

First cover some classic models of network effects, then local network effects.

Key concepts:

- **Externalities:** your action affects others' payoffs.
- **Complementarities:** incentive to take an action depends on others' actions.
- **Tipping points:** thresholds where optimal action switches.
- **Lock-in:** actions of first few agents determine everyone's optimal actions.

Markets with and without Network Effects

Let's start with a simple example of a market without network effects.

Society consists of a large number of individuals i.

- For example, take $i \in [0, 1]$.

Each individual chooses between two products: $s_i \in \{0, 1\}$.

Individuals differ according to a "taste parameter" $x_i \in \mathbb{R}_+$, which measures one's personal preference for product 1 over product 0.

- Assume x_i has some distribution F in the population, with continuous density f.

Individual i's payoff from choosing product s_i is

$$u(s_i, x_i) = (x_i - c)\, s_i,$$

where c is the cost of choosing product 1 over product 0.

- c is common to all players.
  E.g. it could be the price of the good.

Equilibrium without Network Effects

Clearly, if individual i has $x_i > c$, she will choose $s_i = 1$.

- If $x_i < c$, she will choose $s_i = 0$.
- If $x_i = c$, she's indifferent.
  (Doesn't matter what she chooses, as continuous density of f implies almost no one is exactly indifferent.)

Theorem

In the unique equilibrium, fraction $S = 1 - F(c)$ of the population chooses $s_i = 1$.

- Individuals who value product 1 above its cost choose product 1; the rest choose product 0.
- This is just Econ 101.

Introducing Network Effects

Now suppose that $s_i = 1$ corresponds to choosing a new product, like Blu-Ray vs. DVD, or signing up to a new website, like Facebook vs. MySpace.

Assume the utility of using the new product is greater when more people use it.

- This is called a **positive consumption externality** or a **positive network effect**.
- Similar to the Morris contagion model we saw earlier in the class, except now an individual cares about the share of the entire population using the new product. (So like the Morris model where the network is the complete graph.)

Now i's payoff from choosing s_i depends on the share of the population choosing $s_i = 1$, which we denote by S: let

$$u\left(s_i, x_i, S\right) = \left(x_i h\left(S\right) - c\right) s_i,$$

where $h : [0, 1] \to \mathbb{R}$ is a continuous, increasing function capturing the network effect.

Equilibrium with Network Effects

Introducing externalities/network effects turns the market into a game: now each individual must take into account what others are doing when making her own decision.

Since each individual is "small," she cannot affect S, so she takes S as given when making her own decision.

Hence, individual i chooses $s_i = 1$ iff

$$x_i h(S) > c \iff x_i > \frac{c}{h(S)}.$$

The fraction of individuals choosing $s_i = 1$ must be

$$S = 1 - F\left(\frac{c}{h(S)}\right)$$

An equilibrium is therefore a value of S that solves this equation.

Equilibrium with Network Effects (cntd.)

Does an equilibrium always exist?

That is, is there always a solution to

$$\underbrace{S}_{\text{share who adopt}} = \underbrace{D(S)}_{\text{share who want to adopt when } S \text{ others adopt}},$$

where we have seen that

$$D(S) = 1 - F\left(\frac{c}{h(S)}\right).$$

- Left-hand side is continuous in S, ranges from 0 to 1.
- Right-hand side is continuous in S (as h and F are continuous functions) and is always in between 0 and 1.
- Hence, an equilibrium exists by the intermediate value theorem.

Equilibrium with Network Effects (cntd.)

Is the equilibrium always unique?

That is, is there always a unique solution to $S = D(S)$?

- Left-hand side is the 45° line.
- Right-hand side is increasing and non-linear in S.
- Equilibria are the intersections.

Can easily have multiple equilibria.

- This is a key feature of markets with network effects.
- Since I'm more willing to buy the product if others are buying it, there can often be both an equilibrium where few people buy and an equilibrium where many people buy.
- Another instance of multiple equilibria in coordination-like game.

Example

Suppose $h(S) = S$ for all S, and suppose F is the uniform distribution.

Then

$$D(S) = 1 - F\left(\frac{c}{h(S)}\right) = 1 - \min\left\{\frac{c}{S}, 1\right\} = \max\left\{1 - \frac{c}{S}, 0\right\}.$$

Hence, $S = 0$ is an equilibrium, and so is any solution to

$$S = 1 - \frac{c}{S}.$$

This is a quadratic with solutions

$$S = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$

If $c < \frac{1}{4}$, there are three equilibria: $S = 0$, and two equilibria with $S \in (0, 1)$.

Conditions for Multiple Equilibria

In general, the equilibrium values of S are the ones that satisfy $S = D(S)$.

Theorem

Assuming that $c > 0$ *and* $h(0) = 0$ *(so that* $S = 0$ *is always an equilibrium), there are multiple equilibria if and only there exists* $S > 0$ *such that* $S \leq D(S)$.

Real vs. Pecuniary Externalities

We have already discussed how network effects are a form of externality: each agent's action directly affects the payoff of others.

- Externalities turn markets into games.

Aside: in a competitive market, there are always "pecuniary externalities," meaning that each consumer's demand plays a role in determining equilibrium prices.

However, pecuniary externalities cannot lead to inefficient outcomes. This is the **first welfare theorem** of economics.

Real vs. Pecuniary Externalities (cntd.)

If you develop a taste for apples and start buying more of them, this drives up the price of apples, which can make me better-off or worse-off (depending on whether I'm an apple-seller or an apple-buyer.)

- Either way, there's no way to change our behavior to make us both better off.

If you develop a taste for Apple iPhones but previously we were both using Android because we like using the same platform, and now neither of us wants to switch because the other is still using Android, this is an inefficient outcome caused by a real externality.

- We'd both be better off is we changed our behavior.

The difference between competitive markets and markets with network effects is that only the latter involves real externalities (as opposed to only pecuniary externalities).

Externalities and Strategic Complements

The assumption that $h(S)$ is increasing says that we have **positive externalities**: taking $s_i = 1$ increases others' payoffs.

This assumption also implies that the game is one of **strategic complements**: taking $s_i = 1$ increases others' incentives to take $s_i = 1$.

Strategic complementarity is what leads to the possibility of multiple equilibria.

Welfare Comparisons

Important fact: if there are multiple equilibria in a game with strategic complements and positive externalities, they can be **Pareto-ranked**.

- Everyone is better-off in an equilibrium where more people take $s_i = 1$, as compared to one where fewer people take $s_i = 1$.

More precisely, given a group of individuals I, a payoff vector $u = (u_i)_{i \in I}$ is said to **Pareto-dominate** $u' = (u'_i)_{i \in I}$ if

$$\begin{aligned} u_i &\geq u'_i \text{ for all } i \in I \\ u_i &> u'_i \text{ for some } i \in I. \end{aligned}$$

We also say that strategy profile s Pareto-dominates s' if $u(s)$ Pareto-dominates $u(s')$.

Welfare Comparisons (cntd.)

Theorem

In our model of a market with network effects, if S and S' are both equilibrium levels of consumption of product 1 and $S > S'$, then S Pareto-dominates S'.

Proof:

- A type-x_i consumer who purchases gets payoff $x_i h(S) - c$. A consumer who doesn't purchase gets payoff 0.
- When S increases, the purchase option gets better; the no-purchase option stays the same.
- Since consumers with each type x_i choose optimally, every consumer-type is at least weakly better-off when S is higher.
- All consumer-types that purchase at the higher equilibrium consumption level are strictly better-off.

Comparative Statics

In economic models, we are often interesting in **comparative statics**, which tell us how changes in the parameters/environment/network structure affect behavior.

For example, how does an increase in the value of the product change the fraction of agents adopting it?

In general, these questions can be hard to answer in models with multiple equilibria: if there's some set of equilibria at one parameter value and another set at another value, how do we compare them?

- (What does it mean for one *set* to be higher than another?)

However, in games with strategic complements, we can often analyze comparative statics by considering the greatest equilibrium (or the smallest).

Comparative Statics (cntd.)

Theorem

In our model of a market with network effects, suppose that x_i increases for a positive fraction of agents and remains the same for everyone else. Then the largest and smallest equilibrium values of S both weakly increase (i.e. either increase or stay the same).

Intuition:

- Holding S fixed, the fact that x_i increases for some agents means that some of them may switch to taking $s_i = 1$.
  (The others do not change their actions.)
  This **direct effect** pushes S up.
- Since S increases, this increases the network externality $h(S)$.
  This makes more agents switch to $s_i = 1$.
  This **indirect effect** pushes S up even further.

Comparative Statics (cntd.)

Theorem

In our model of a market with network effects, suppose that x_i increases for a positive fraction of agents and remains the same for everyone else. Then the largest and smallest equilibrium values of S both weakly increase (i.e. either increase or stay the same).

Visual/mathematical argument:

- Recall that the equilibria are the intersections of the curve $D(S)$ with the 45° line.
- Increasing x_i for a positive fraction of agents corresponds to shifting the curve $D(S)$ up.
- This always weakly increases its greatest and smallest intersection with the 45° line.

Note: intermediate equilibria can shift the other way.

- Should we worry about this?

Stability and Tipping

To understand whether we should worry about intermediate equilibria, we need to understand which equilibria are most likely to emerge when there are multiple equilibria.

It's useful here to think about the **dynamics** of how a market is likely to come to reach an equilibrium.

If fraction S of the population is taking action 1, then fraction $D(S)$ would like to take action 1 in response.

Hence, if the population finds itself at a point where $D(S) > S$, the share taking action 1 will increase.

Similarly, if $D(S) < S$, the share taking action 1 will decrease.

This simple dynamic process is called **best-response dynamics**.

Stability and Tipping (cntd.)

Say that $D(S)$ cross the 45° line at S **from below** if it's below the 45° line for S' slightly below S, and above the 45° line for S' slightly above S.

Equilibria where $D(S)$ cross the 45° line from above are **unstable** (sometimes called **tipping points**).

- If we slightly decrease S, then best-response dynamics will bring S further down.
- If we slightly increase S, then best-response dynamics will bring S further up.

Stability and Tipping (cntd.)

Theorem

In our model of a market with network effects, the largest and smallest equilibrium values of S are both stable.
Intermediate equilibrium values may be unstable.

More generally, it can be shown that the stable equilibria are precisely those where the comparative static response to a "small" change in parameters goes the "right way" (i.e., the same way as the direct effect, as well as the greatest and smallest equilibria).

- This is called the **correspondence principle** (due to Paul Samuelson): at a stable equilibrium, local comparative statics "correspond" to best-response dynamics. (They go the same way.)

A Related Idea: Lock-In

Suppose we instead consider a dynamic version of the model, where

- consumer arrive over time,
- each consumer chooses $s_i = 0$ or 1 once-and-for-all when she arrives,
- a consumer of type x_i who enters the market at time t and chooses s_i gets payoff $(x_i h(S_t) - c) s_i$, where S_t is the share of the population that has chosen $s_i = 1$ **as of time t.**

Over time, S_t will converge to one of the **stable** equilibrium values.

- Can't converge to an unstable equilibrium, because it drifts away from this point whenever it is not exactly equal to it.
- Converges somewhere by law of large numbers.

Which stable equilibrium value is ultimately reached depends on whether early consumers tend to have high or low x_i, which is a matter of chance.

Lock-In: Examples

If we interpret 0 and 1 as two different products, this says that which product "wins" the market is determined by the random tastes of early movers.

Examples:

- QWERTY keyboard
- VHS vs. Betamax
- BitCoin vs. Ethereum
- MySpace vs. Facebook?

Summary of Network Effects

- An environment exhibits **network effects** if each individual's optimal action depends on some average of others' actions.
- Markets with network effects can have multiple equilibria.
- In games with strategic complements and positive externalities, equilibria are Pareto-ranked: equilibria with higher actions are better for everyone.
- In games with strategic complements, comparative statics for the largest and smallest equilibria are easy to analyze. For example, they move the "right way" in response to shifts in demand.
- These equilibria are also stable under best-response dynamics.

Local Network Effects

Our model of a market with network effects features strategic complementarity.

- This gives the game a coordination aspect.

A version of the model where each player cares about the share of her **neighbors** who take $s_i = 1$ rather than 0 coincides with the Morris contagion model we've already considered.

- The Morris model is thus a model of local network effects.

Today we study two other local network effect models:

- residential choice
- linear best-response games

Residential Choice

One of the earliest and still most important models of local network effects was Schelling's (1972) model of segregation.

Schelling was writing at a time when the Jim Crow laws in the US South and other forms of organized, coercive segregation were breaking down, but racial segregation remained extremely high.

There were (are) many possible explanations for high levels of racial segregation, but Schelling highlighted a force that's particularly insidious and suggests that reducing segregation requires much more than eliminating blatant discrimination like Jim Crow/redlining: a slight individual preference for living near people like yourself can lead to dramatically high levels of equilibrium segregation due to "tipping point" effects.

- Individuals don't have to want to live in a large local majority (e.g. an overwhelmingly white neighborhood if white or an overwhelmingly black neighborhood if black) for extreme segregation to arise. It's enough that they don't want to be in a small local minority.

Residential Choice (cntd.)

Schelling considered the following simple model:

- Agent of two types (race, income, education, political party affiliation, etc.) randomly arranged on a grid, with a few empty spaces.
- An agent is **satisfied** if at least fraction p of her (up to 8) neighbors are of her own type, **unsatisfied** otherwise.
- Each period, one unsatisfied agent is chosen at random. She moves to the closest empty space where she'll be satisfied.
- The process continues until everyone is satisfied (i.e., no one wants to move).

Residential Choice (cntd.)

What happens?

- Suppose by chance one part of the grid starts out slightly majority black.
- If p is relatively small, most whites in this area will be satisfied, but some (the ones who by chance have very few white neighbors) will be unsatisfied.
- The unsatisfied whites move out, and blacks move in.
- This makes more whites unsatisfied, so the process continues, potentially leading to extreme segregation.

The model is hard to study analytically (a Markov chain on a very large state space), but experimentally Schelling found that if $p = 0.5$ then in equilibrium on average 80% of one's neighbors are of one's own type.

Schelling's segregation model was one of the first **agent-based models**: computational models for simulating interactions among autonomous agents. (Useful complement to theoretical analysis.)

Tipping Points in the Real-World

A 2008 paper by economists Card, Mas, and Rothstein ("Tipping Points and the Dynamics of Segregation") empirically analyzes tipping-point behavior in residential segregation in the US from 1970 to 2000.

They ask how the share of minority residents in a census tract in 1970 predicts the change in this share between 1970 and 2000.

- They also ask how the relationship between *share(1970)* and *share(2000)-share(1970)* varies across different cities.

They find that census tracts with $<\approx$10% minorities in 1970 experienced a decrease in minority population share between 1970 and 2000, while tracts with $>\approx$10% minorities experienced an increase in minority population share.

They also find that different cities had different "tipping points" (range from $\approx$5% to $\approx$20%), and that this correlates with racial attitudes among white residents, with cities with more "racially tolerant" whites having higher tipping points.

Game Theory of Local Network Effects: Linear Best Response Games

The Morris contagion model and the Schelling segregation model are important examples of models of local network effects, based on different kinds of desires for "local coordination."

Is there a more general way to think about the game theory of local network effects that captures both local strategic complements (i.e. own optimal action is increasing in neighbors' actions) and local strategic substitutes (i.e. own optimal action is decreasing in neighbors' actions)?

In general, such problems can be very complicated.

But an important and tractable special case arises when each individual's optimal strategy (i.e. best response) is a **linear** function of her opponent's actions.

We consider first strategic substitutes, then complements.

Examples of Local Strategic Complements and Substitutes

Aside: (local) strategic complements vs. substitutes in an important concept for understanding strategic interactions on networks.

- Captures a key aspect of many strategic interactions: whether I want to increase or decrease my "activity level" when my neighbors do so.

Strategic complements = "coordination game flavor."

- Spend more time on an app or using another technology when your friends do.
- Engage more in various good or bad behaviors when your friends do (e.g., even committing crimes).

Strategic substitutes = "anti-coordination game flavor."

- Work less hard on providing a public good when others pick up the slack (e.g., learning about new technologies, cleaning your dorm room, working on psets?).

Games on Networks with 1-Dimensional Actions

n players on an undirected network with adjacency matrix g

- $g_{ij} \in \{0, 1\}$, $g_{ij} = 1$ means i and j are linked.

Consider the static game where each player i takes a non-negative, real-valued action $x_i \geq 0$.

- **Interpretation:** x_i is player i's "activity level," e.g. how much effort she exerts on some project.
- Let $x = (x_1, \ldots, x_n)$ denote a strategy profile.

Payoff Functions

Assume payoff functions take the form

$$u_i(x) = b_i\left(x_i + \delta \sum_{j \neq i} g_{ij} x_j\right) - c_i x_i,$$

where

- $b_i(\cdot)$ is a "benefit function," assumed to be differentiable, increasing, and concave.
- δ is a parameter measuring the strength of strategic complements or substitutes.
  - We'll see that $\delta < 0$ is strategic complements, $\delta > 0$ is strategic substitutes.
- $c_i > 0$ is a "cost parameter," measuring the cost of effort.

Strategic Substitutes/Complements

Why does $\delta > 0$ correspond to strategic substitutes?

First-order condition is

$$\underbrace{b_i'\left(x_i + \delta \sum_{j \neq i} g_{ij} x_j\right)}_{\text{marginal benefit of effort}} = \underbrace{c_i}_{\text{marginal cost of effort}}$$

Since b_i is concave, b_i' is decreasing, so as x_j increases (for j such that $g_{ij} = 1$), x_i decreases at rate δ.

Similarly, $\delta < 0$ corresponds to strategic complements.

Best Responses

What is player i's best response to opponents' actions x_{-i}?

First-order condition is

$$\underbrace{b_i'\left(x_i + \delta \sum_{j \neq i} g_{ij} x_j\right)}_{\text{marginal benefit of effort}} = \underbrace{c_i}_{\text{marginal cost of effort}}$$

Let $\bar{x}_i$ be the solution to

$$b_i'\left(\bar{x}_i\right) = c_i.$$

Then i's best response to x_{-i} is given by

$$x_i^* = \begin{cases} \bar{x}_i - \delta \sum_{j \neq i} g_{ij} x_j & \text{if } \delta \sum_{j \neq i} g_{ij} x_j \leq \bar{x}_i \\ 0 & \text{otherwise} \end{cases}$$

Note: x_i^* is **linear** in x_j (for $j \in N_i$), and is **increasing** if $\delta < 0$ (complements) and **decreasing** if $\delta > 0$ (substitutes).

Example: Local Public Goods (\delta=1)

- Suppose the network represents a small town.
- Suppose x_i measures how much effort agent i puts into maintaining her garden.
- Suppose $\delta = 1$ and $\bar{x}_i = 1$ for all i: each agent is willing to work on her garden up to the point where the sum of the nicenesses of her garden and her neighbors' gardens equals 1 (beyond that point gardening still provides benefits, but the local neighborhood is already nice enough that the benefits are not worth the cost of effort).
- A (pure-strategy) Nash equilibrium is a vector x such that, for all i, $x_i + \sum_{j \neq i} g_{ij} x_j = 1$.

This is a leading special case of strategic substitutes with some nice features, so we'll spend a few minutes on it.

Local Public Goods (cntd.)

What do PSNE look like in this model?

One special case: each agent i takes either $x_i = 0$ or $x_i = 1$.

- Each agent either works hard enough to beautify the whole local neighborhood by herself, or completely slacks off.
- Call such an equilibrium **specialized.**

Note: it can be shown that only specialized equilibria can be stable under best-response dynamics.

- **Intuition:** if two neighbors are both taking actions strictly in between 0 and 1, then if one of them slightly increases her action, the other will decrease his action, so the first one will increase her action further, etc.

Local Public Goods (cntd.)

What do specialized equilibria look like?

It turns out they can be related to the graph-theoretic concept of a **maximal independent set.**

A set of nodes $S \subseteq N$ in a network is a **maximal independent set** if no two nodes in S are linked to each other, and in addition every node not in S is linked to at least one node in S.

E.g. In a star network, there are exactly two maximal independent sets. What are they?

Local Public Goods (cntd.)

Theorem

For any $S \subseteq N$, there is a specialized equilibrium where everyone in S takes $x_i = 1$ and everyone else takes $x_i = 0$ if and only if S is a maximal independent set.
In particular, a specialized equilibrium always exists.

Proof:

- If everyone else takes action $x_j \in \{0, 1\}$, player i's optimal action is $x_i = 1$ if she's not linked to anyone who takes $x_j = 1$, and is $x_i = 0$ if she is.
- Hence, S corresponds to a specialized equilibrium iff everyone in S is not linked to anyone else in S (so 1 is optimal for agents in S), and everyone else is linked to someone in S (so 0 is optimal for agents outside S).
- This is precisely the definition of a maximal independent set.

\delta<1

When $\delta < 1$, finding equilibria is somewhat more challenging, but it still just involves solving a relatively simple system of linear equations.

E.g. consider a star network, with one center node and 3 periphery nodes.

- It's a PSNE for the center to play $x_C > 0$ and the peripheral agents to all play $x_P > 0$ if and only if

$$\begin{aligned} x_C + 3\delta x_P &= 1 \\ x_P + \delta x_C &= 1 \end{aligned}$$

- It's a PSNE for the center to play $x_C = 0$ and the peripheral agents to all play $x_P > 0$ if and only if

$$\begin{aligned} 3\delta x_P &\geq 1 \\ x_P &= 1 \end{aligned}$$

- There is never a PSNE where the peripheral agents play $x_P = 0$. (Why not?)

Finding all PSNE

In general, to find all the PSNE,

1. Consider separately each possible set of players who could take $x_i > 0$.
2. For each such set, see if there is indeed an equilibrium where exactly this set takes $x_i > 0$ by solving the resulting system of linear equations, checking that all elements of the solution are positive, and finally checking that it is a best response for everyone else to take $x_i = 0$.

Linear-Quadratic Payoffs

A closely related linear best-response game arises when players have linear-quadratic payoffs:

$$u_i\left(x\right) = \bar{x}_i x_i - \delta \sum_{i,j} g_{ij} x_i x_j - \frac{1}{2} x_i^2$$

where now $\bar{x}_i$ is an arbitrary constant.

The first-order condition is

$$x_i = \bar{x}_i - \delta \sum_{j \neq i} g_{ij} x_j .$$

Hence, each player i has exactly the same best response function as before:

$$x_i^* = \begin{cases} \bar{x}_i - \delta \sum_{j \neq i} g_{ij} x_j & \text{if } \delta \sum_{j \neq i} g_{ij} x_j \leq \bar{x}_i \\ 0 & \text{otherwise} \end{cases}$$

This implies that the equilibria of this game are identical to those of the previous one.

Connection to Potential Games

It turns out that the linear-quadratic game is a potential game, with potential function

$$\phi\left(x\right) = \sum_i \left(\bar{x}_i x_i - \frac{1}{2}x_i^2\right) \Bigg(- \frac{1}{2}\delta \sum_{i,j} g_{ij} x_i x_j$$

- This is the sum of everyone's payoffs, but with the externality term weighted by $\frac{1}{2}$.

To see that this is indeed a potential function, take the first-order condition for maximizing the potential with respect to x_i:

$$x_i = \bar{x}_i - \delta \sum_{j \neq i} g_{ij} x_j .$$

- Same as first-order condition for x_i to be optimal for player i.
- Note: we had to weight the externality term by $\frac{1}{2}$ because it shows up once for (i, j) and once for (j, i).

Connection to Potential Games (cntd.)

As we saw last week, being a potential game is a useful property.

- Implies that a PSNE exists. Hence, every linear best-response game has a PSNE.
- If can also be used to give conditions for other properties, like uniqueness and stability of PSNE. We skip these results.

Strategic Complements

Last topic: linear-quadratic games with strategic **complements**.

Same setup as before, but now payoff function is

$$u_i(x) = x_i + \delta \sum_{j \neq i} g_{ij} x_i x_j - \frac{1}{2} x_i^2.$$

- We have just set the $\bar{x}_i$'s equal to 1 and flipped the sign on the externality term.

First-order condition for x_i:

$$x_i = 1 + \delta \sum_{j \neq i} g_{ij} x_j$$

- Each player wants to do 1 unit of activity "by themself," plus another δ units for each unit of activity of their neighbors.

Equilibrium with Strategic Complements

First-order conditions:

$$x_i = 1 + \delta \sum_{j \neq i} g_{ij} x_j$$

Stacking this equation for all $i \in N$ yields

$$x = \mathbf{1} + \delta G x,$$

where G is the adjacency matrix and $\mathbf{1}$ is the vector of 1's.

To express the vector of equilibrium actions in closed form, re-write this as

$$(I - \delta G)\, x = \mathbf{1}.$$

If the matrix $(I - \delta G)$ is invertible, can solve in closed form as

$$x = (I - \delta G)^{-1}\, \mathbf{1}.$$

- Leontief inverse again!

Equilibrium and Katz-Bonacich Centrality

Recall from Lecture 3 that the Katz-Bonacich centrality of node i in an undirected network G (one of the ways we discussed of measuring the importance or "prestige" of a node) is given by

$$C_i = 1 + \frac{1}{\lambda} \sum_{j \neq i} g_{ij} C_j,$$

or in vector form

$$C = \mathbf{1} + \frac{1}{\lambda} G C.$$

Interpretation: each node receives 1 unit of prestige for free, plus another $1/\lambda$ units for each unit of prestige of their neighbors.

Equilibrium and Katz-Bonacich Centrality (cntd.)

Katz-Bonacich centrality:

$$C = \mathbf{1} + \frac{1}{\lambda} G C.$$

Letting $\lambda = \frac{1}{\delta}$, this is exactly the equation for the vector of equilibrium actions in a linear-quadratic game with strategic complements:

$$x = \mathbf{1} + \delta G x.$$

Thus, a player's equilibrium action is the same as their Katz-Bonacich centrality!

- Your action in the game is determined precisely by how "central" you are in the network.
- Centrality and equilibrium actions are perfectly correlated: more central players take higher actions.

Explanation

Katz-Bonacich centrality: each node receives 1 unit of prestige for free, plus another $1/\lambda$ units for each unit of prestige of their neighbors.

Equilibrium actions: each player does 1 unit of activity "by themself," plus another δ units for each unit of activity of their neighbors.

This makes it obvious that Katz-Bonacich centrality and equilibrium actions are measuring the same thing.

Summary of Local Network Effects

- Local network effects can explain phenomena with both spatial and economic aspects, like segregation and local public good provision.
- The Schelling segregation model shows that fairly mild homophily can sometimes lead to extreme segregation.
- Linear best-response games are a tractable way to model local strategic substitutes or complements, with connections to both potential games and centrality measures.