Problem Set 4 Solutions

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1 Walrasian Equilibrium with Production

There is an economy with 2 goods and I consumers. Each consumer i has the following utility function:

$$u^i(x_1^i, x_2^i) = x_1^i + \log x_i^2$$

Each consumer starts with an endowment of 4 units of good 1 and none of good 2. Good 2 can be produced from good 1 using the following production function:

$$y_2 = \sqrt{z_1}$$

where z_1 is the amount of good 1 used as an input and y_2 the output of good 2 produced. There is a single firm that produces good 2 from good 1 and each consumer owns an equal share of this firm, so they each get an equal share of profits.
The price of good 1 is normalized to 1. Let the price of good 2 be p. profits.

a) Write down the conditions that a price p and an allocation $(\{x_1^i\}_{i=1}^I, \{x_2^i\}_{i=1}^I, y_2, z_1)$ must satisfy to be a Walrasian equilibrium.

Solution: Firstly, the firm should maximize profits:

$$(y_2, z_1) = \arg\max py_2 - z_1 \text{ s.t. } y_2 = \sqrt{z_1}$$

Secondly, the allocations of the xs should maximize utility given the consumer's budget (which includes their share of the profits):

$$x^i = (x_1^i, x_2^i) = \arg\max u^i(x^i) \text{ s.t. } x_1^i + px_2^i \leq 4 + \frac{1}{I}(py_2 - z_1) \ \ \forall i$$

Thirdly, markets must clear:

$$\sum_i x_1^i + z_1 = 4I$$

$$\sum_i x_2^i = y_2$$

b) Solve the firm's profit maximization problem for an arbitrary p. What is its profit function, $\pi(p)$?

Solution: We get that supply is $y_2(p) = \frac{p}{2}$. Profits are

$$\pi(p) = \frac{p^2}{4}$$

c) Solve the consumer's utility maximization problem for an arbitrary p.

Solution: Using our value of $\pi(p)$ from b), we get

$$x_2^i(p) = \frac{1}{p}$$

$$x_1^i(p) = 3 + \frac{p^2}{4I}$$

d) Using your answers to b) and c) and market clearing, find the Walrasian equilibrium.

Solution: The equilibrium price works out as $\sqrt{2I}$ (it's easiest to use the market clearing condition for good 2) and the allocations are then

$$x_2^i = \frac{1}{\sqrt{2I}} \quad \forall i$$

$$x_1^i = 3.5 \quad \forall i$$

$$z_1 = \frac{I}{2}$$

$$y_2 = \sqrt{\frac{I}{2}}$$

e) Calculate each consumer's utility in equilibrium. How does it depend on I? Can you give any intuition for this result?

Solution: Utility works out as

$$3.5 - \frac{1}{2}\log(2I)$$

It is decreasing in I. Giving intuition for this was actually not an easy question at all. As I increases, the total demand for good 2 at given p increases proportionally. Supply of good 1, from the endowments, also increases proportionally. But the production technology has decreasing returns to scale, i.e. we are getting less efficient at converting good 1 to good 2. So the price of good 2 must rise to clear the market, which lowers consumers' demand for good 2 and therefore their utility.

2 Trade and the 2x2x2 Model

Suppose there are two goods x, y, two factors K, L, and countries A, B have the same preferences and technology. Preferences follow the function

$$U_i(x_i, y_i) = 2\sqrt{x_i} + 2\sqrt{y_i}$$

and production technology is Leontief with good x following the production function

$$x = f(K_x, L_x) = \min\{2K_x, L_x\}$$

and good y following the production function

$$y = g(K_y, L_y) = \min\{K_y, L_y\}$$

Country A has endowments $\omega_A = (\overline{K_A}, \overline{L_A}) = (20, 30)$, and country B has endowments $\omega_A = (\overline{K_A}, \overline{L_A}) = (35, 50)$. The prices paid for capital and labor are r and w respectively.

a) Suppose the world is in autarky. We will go through the steps to solve for equilibrium prices (p_x^c, p_y^c, r^c, w^c), consumption, and factor allocations in the two countries.

i) We will find the equilibrium conditions for the production side first. What are the unit-cost functions for production of the two goods?

Solution: To produce one unit of good x, the cost function is

$$c_x(r, w) = \frac{1}{2}r + w$$

where the input combination is $a_{Kx}(r, w) = \frac{1}{2}, a_{Lx}(r, w) = 1$ and is fixed regardless of factor prices. And similarly, the cost function for y is

$$c_y(r, w) = r + w$$

with the input combination $a_{Ky}(r, w) = 1, a_{Ly}(r, w) = 1$

ii) Which good is more capital-intensive?

Solution: Good y is more capital-intensive since its capital-intensity ratio is higher than for good x.

iii) Notice that the production function is CRS and therefore profits are zero. What are the equations determining prices p_x, p_y as a function of r and w?

Solution: With zero profits, then it must be that $c_x(r, w) = p_x$ and that $c_y(r, w) = p_y$. Therefore prices are $p_x = \frac{1}{2}r + w$, $p_y = r + w$.

iv) From firm optimization and market clearing for the factor endowments for each country, calculate the factor allocations $K_x^c, L_x^c, K_y^c, L_y^c$ for each country $c \in \{A, B\}$. Calculate total production for goods x^c, y^c.

Solution: Countries A, B have different factor endowments and this will affect production. Equilibrium factor allocations are such that

$$\frac{K_x}{L_x} = \frac{a_{Kx}}{a_{Lx}} = \frac{1}{2}$$

$$\frac{K_y}{L_y} = \frac{a_{Ky}}{a_{Ly}} = 1$$

And market clearing for factors in country A means $K_x^A + K_y^A = 20$, $L_x^A + L_y^A = 30$. From these sets of equation, we can solve for $K_x^A = 10, K_y^A = 10, L_x^A = 20, L_y^A = 10$.

Using the production function, then total production (and consumption) of goods x and y are $x^A = 20$, $y^A = 10$.

Market clearing for factors in country B means $K_x^B + K_y^B = 35$, $L_x^B + L_y^B = 50$. We can solve for $K_x^B = 15, K_y^B = 20, L_x^B = 30, L_y^B = 20$.

Using the production function, then total production (and consumption) of goods x and y are $x^B = 30$, $y^B = 20$.

v) Now we move to the consumer side. From utility maximization (assume an interior solution), what must the price-ratio equal?

Solution: Prices are determined from consumer demand. The price ratio follows

$$\frac{MU_x}{MU_y} = \left(\frac{y}{x}\right)^{1/2} = \frac{p_x}{p_y}$$

vi) We find the general equilibrium by putting the production and consumer conditions together. Using the zero-profit equations and the utility maximization condition, solve for equilibrium prices (p_x^c, p_y^c, r^c, w^c).

Hint: Use Walras' Law and normalize wage $w^c = 1$.

Solution: The set of equilibrium conditions for country A are

$$\left(\frac{1}{2}\right)^{1/2} = \frac{p_x}{p_y}$$

from utility maximization and

$$p_x^A = \frac{1}{2}r^A + w^A$$
$$p_y^A = r^A + w^A$$

from zero-profit. Solving for prices gives $r^A \approx 1.414$, $p_x^A \approx 1.707$, $p_y^A \approx 2.414$

The set of equilibrium conditions for country B are

$$\left(\frac{2}{3}\right)^{1/2} = \frac{p_x}{p_y}$$

$$p_x^B = \frac{1}{2}r^B + w^B$$
$$p_y^B = r^B + w^B$$

This leads to the prices $r^B \approx 0.58, p_x^B \approx 1.29, p_y^B \approx 1.58$

b) Suppose now there is free trade. We will go through the steps to solve for equilibrium prices, consumption, factor allocations, and net exports from A to B.

i) With free trade and the same technology and preferences in both countries, what does this tell us about good and factor prices in country A compared to country B?

Solution: With free trade, goods can be sold across countries that have the same preferences, so prices must be the same across the world. Since both countries have the same technology, then factor prices must also be the same across the world (FPE Theorem).

ii) Using your answer in (i), solve for good and factor prices.

Solution: The set of equations that determine prices are

$$\frac{MU_x}{MU_y} = \left(\frac{y}{x}\right)^{1/2} = \frac{p_x^*}{p_y^*}$$

$$p_x^* = \frac{1}{2}r^* + w^*$$

$$p_y^* = r^* + w^*$$

Total production in the world is $x^W = x^A + x^B = 50$, $y^W = y^A + y^B = 30$, and therefore the world price ratio is

$$\frac{p_x^*}{p_y^*} = \left(\frac{y^w}{x^w}\right)^{1/2} = \left(\frac{30}{50}\right)^{1/2}$$

Normalize $w^* = 1$. The rest of the prices are $r^* \approx 0.821, p_x^* \approx 1.41, p_y^* \approx 1.821$

iii) Do factor allocations and production change with free trade? How about consumption?

Solution: Factor allocations and production for both countries do not change with free trade. However, consumption can change since countries can buy goods from each other.

iv) Calculate net exports from A to B for goods x and y. *Hint:* calculate total income for each country, then solve for consumption using the good prices.

Solution: Total income for country A is $p_x^*(x^A) + p_y^*(y^A) \approx 1.41(20) + 1.821(10) \approx 46.41$. (You should get the same result using factor endowments and factor prices).

Consumption must satisfy

$$\frac{MU_x}{MU_y} = \left(\frac{y_c^A}{x_c^A}\right)^{1/2} = \left(\frac{30}{50}\right)^{1/2} \Rightarrow \frac{y_c^A}{x_c^A} = \frac{3}{5}$$

subject to the budget constraint $p_x^*(x_c^A) + p_y^*(y_c^A) = 1.41(x_c^A) + 1.821(y_c^A) = 46.41$. Consumption solutions are $x_c^A \approx 18.5, y_c^A \approx 11.1$

Total income for country B is $p_x^*(x^B)+p_y^*(y^B) \approx 1.41(30)+1.821(20) \approx 78.72$. We again have

$$\frac{y_c^A}{x_c^A} = \frac{3}{5}$$

subject to the budget constraint $p_x^*(x_c^B)+p_y^*(y_c^B) = 1.41(x_c^B)+1.821(y_c^B) = 78.72$. Consumption solutions are $x_c^B \approx 31.5, y_c^A \approx 18.9$

Net exports from A to B are then $x^{EXP} = 1.5$, $y^{EXP} = -1.1$

c) Interpret your results from c) in terms of the Heckscher-Ohlin theorem.

Solution: The results for trade follow exactly the H-O theorem. Country B is relatively more capital abundant compared to country A, and therefore it exports y, which is more capital intensive. Country A is relatively more labor abundant, and it exports good x, which is more labor-intensive.