# Problem Set 1 Solutions

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## 1 Problem 1: Preference Relations and Utility Functions

**a)** Let $X = \mathbb{R}^2_+$ and there be two points $x = (x_1, x_2)$, $y = (y_1, y_2)$.
Suppose $x \succeq y$ if $x_1 > y_1$ or if $x_1 = y_1$ and $x_2 \geq y_2$.
Is the preference relation $\succeq$ complete? Transitive? Why or why not?

**Solution**: These preferences are called lexicographic preferences.
Completeness: If $x_1 > y_1$ ($y_1 > x_1$), then $x \succeq y$ ($y \succeq x$). If $x_1 = y_1$, then either $x_2 > y_2$ so that $x \succeq y$, or $y_2 > x_2$ so that $y \succeq x$, or $x_2 = y_2$ so that $x \succeq y$ and $y \succeq x$.
Transitivity: Let $x, y, z \in \mathbb{R}^2_+$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$. Suppose that $x \succeq y$ and $y \succeq z$. Then we want to show that $x \succeq z$. As we assume that $x \succeq y$ then either

$$x_1 > y_1 \tag{1}$$
$$x_1 = y_1 \text{ and } x_2 \geq y_2 \tag{2}$$

As we assume that $y \succeq z$ then either

$$y_1 > z_1 \tag{3}$$
$$y_1 = z_1 \text{ and } y_2 \geq z_2 \tag{4}$$

Now we show that $x \succeq z$.
If (1) and (3) are true, then $x_1 > z_1$ and therefore $x \succeq z$.
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If (2) and (4) are true, then $x_1 = z_1$ and $x_2 \geq z_2$ and therefore $x \succeq z$
So in all possible cases, $x \succeq z$ as required.

**b)** John has preferences over consumption bundles $(A, B) \in \mathbb{R}^2_+$ characterized by utility function $U(A, B) = A^{\frac{1}{3}} B^{\frac{2}{3}}$. Show that John's preferences satisfy strict monotonicity, local non-satiation, strict convexity, and continuity.

**Solution**: Strict monotonicity: Follows from the fact that $U(A, B)$ is strictly increasing in both $A$ and $B$.
Local non-satiation: Follows from the fact that the gradient of $U(A, B)$ is never the zero vector on $\mathbb{R}^2_+$.
Strict convexity: Emphasis on strict. If preferences were linear, they would

be weakly convex. Preferences are strictly convex if the utility function is strictly quasi-concave. Strict concavity implies strictly quasi-concave, and to show that the utility function is strictly concave, we show that the Hessian of the utility function is negative definite.

$$H = \begin{bmatrix} -\frac{2}{9}A^{-5/3}B^{2/3} & \frac{2}{9}A^{-2/3}B^{-1/3} \\ \frac{2}{9}A^{-2/3}B^{-1/3} & -\frac{2}{9}A^{1/3}B^{-4/3} \end{bmatrix}$$

The first principal minors $-\frac{2}{9}A^{-5/3}B^{2/3}$ and $-\frac{2}{9}A^{1/3}B^{-4/3}$ are both negative, and the second principal minor (the determinant) is positive.
Continuity: $U(A, B)$ is continuous.

**c)** Consider the following constrained maximization problem using the utility function from part b)

$$\begin{aligned} \max \quad & U(A, B) = A^{\frac{1}{3}}B^{\frac{2}{3}} \\ \text{s.t.} \quad & p_A A + p_B B \leq I \\ & A \geq 0 \text{ and } B \geq 0 \end{aligned}$$

where $p_A, p_B, I > 0$. Let $A^*, B^*$ denote the solution to the above problem.

**i.** Can we ever have $A^* = 0$ or $B^* = 0$? Why or why not?

**Solution**: No we cannot. If $A^* = 0$ or $B^* = 0$, then $U(A^*, B^*) = 0$. However since $I > 0$ and prices $p_A, p_B > 0$, then there exists $A', B' > 0$ where $p_A A + p_B B \leq I$ and $U(A', B') > 0$, which contradicts that $(A^*, B^*)$ was optimal.

**ii)** Can we ever have $p_A A^* + p_B B^* < I$? Why or why not?

**Solution**: No. This is because preferences are strictly monotonic in $A$ and $B$. Therefore, if $p_A A^* + p_B B^* < I$, then there exists the point $(A + \epsilon, B + \epsilon)$ where $p_A(A^* + \epsilon) + p_B(B^* + \epsilon) \leq I$ and $U(A + \epsilon, B + \epsilon) > U(A, B)$, which contradicts $A^*, B^*$ being a maximum.

**iii)** Set up the consumer's Lagrangian and find the first-order conditions. How do you know that these first-order conditions are sufficient to characterise the solution to the consumer's problem? For what values of $p_A$, $p_B$ will the consumer consume twice as much $A$ as $B$?

**Solution**: The Lagrangian is $L(A, B) = A^{1/3}B^{2/3} + \lambda(I - p_A A - p_B B)$ and the first-order conditions are

$$\frac{1}{3}A^{-2/3}B^{2/3} - \lambda p_A = 0$$
$$\frac{2}{3}A^{1/3}B^{-1/3} - \lambda p_B = 0$$

These conditions characterize the solution to the consumer's problem because the utility function is concave, the constraint set is convex, and $\lim_{A\to 0} U_A = \infty$, $\lim_{B\to 0} U_B = \infty$. To figure out when the consumer will buy twice as much $A$ as $B$, rearrange the FOCs to set MRS equal to the price ratio. $MRS = \frac{B}{2A} = \frac{p_A}{p_B}$. Then we see that $A = 2B$ whenever $\frac{p_A}{p_B} = 1/4$

## 2 Problem 2: Income and Substitution Effects

A (potential) worker has utility over consumption $c$ and leisure $l$ given by

$$U(c,l) = \alpha\frac{c^\delta}{\delta} + \beta\frac{l^\delta}{\delta}$$

where $\delta < 1$. She has $T$ hours to allocate between leisure and work. For each hour she works, she earns a wage $w$ to spend on consumption $c$, which we normalize the price of $c$ to one. However, because her wife works, she receives an additional 'non-labor income' $Y$ regardless of how much she works. Assume she takes $Y$ as given (i.e. her own decisions do not affect her wife's labor supply). She therefore maximizes utility subject to the following constraints:

$$\begin{aligned}&c \le w(T-l) + Y\\ &c \ge 0\\ &0 \le l \le T\end{aligned}$$

**a)** Without writing down the Lagrangian or solving the optimization problem, identify which constraints above will always bind (hold with equality) at the optimum, and which constraints will always be slack (not hold with equality). Are there any constraints which fall into neither category?

**Solution**: The budget constraint $c \le w(T-l) + Y$ is going to bind. To make clear the link with more standard budget constraints, we can rewrite it as $c + wl \le wT + Y$ (in other words, leisure is 'bought' at the price $w$). Then, because the worker has strictly increasing utility in both $c$ and $l$, we see that she will always want to exhaust her 'budget' $wT + Y$.

The constraints $c \ge 0$ and $l \ge 0$ will always be slack. This is because the consumer's marginal utilities are $c^{\delta-1}$ and $l^{\delta-1}$, which become infinite as consumption and leisure approach zero, so it can't be optimal to consume zero of either.

The constraint $l \le T$ might or might not bind, depending on parameters. Even if the worker chooses $l = T$, i.e. doesn't work at all, she can still consume something because of her nonlabor income $Y$, so if leisure is particularly valuable to her she might choose that.

**b)** Set up the Lagrangian and write out all the relevant conditions for a solution, using your answer to a) to help simplify things.

**Solution:** We can rewrite the problem using a) as

$$\max U(c,l) = \alpha\frac{c^\delta}{\delta} + \beta\frac{l^\delta}{\delta}$$

subject to

$$c = w(T-l) + Y$$
$$l \leq T$$

The associated Lagrangian is

$$L(c,l) = \alpha\frac{c^\delta}{\delta} + \beta\frac{l^\delta}{\delta} - \lambda(c + wl - wT - Y) - \mu(l - T)$$

The conditions for an optimum are firstly the first-order conditions:

$$\frac{dL}{dc} = 0 \ \Rightarrow \ \alpha c^{\delta-1} - \lambda = 0$$

$$\frac{dL}{dl} = 0 \ \Rightarrow \ \beta l^{\delta-1} - w\lambda - \mu = 0$$

and then also the non-negativity constraints on the Lagrange multipliers and the complementary slackness conditions:

$$\lambda \geq 0; \mu \geq 0$$

$$\lambda(c + wl - wT - Y) = 0; \mu(l - T) = 0$$

and finally the constraints themselves:

$$c = w(T-l) + Y$$
$$l \leq T$$

**c)** Assume now that the solutions are at an interior point. How do $c$ and $l$ change as non-labor income $Y$ increases? What does this tell us about whether $c, l$ are normal goods?

**Solution**: First we find the Marshallian demand functions. Assuming an interior solution means that the constraint $l \leq T$ does not bind, and from a) we know the budget constraint $c = w(T-l) + Y$ always binds. The new Lagrangian is

$$L(c,l) = \alpha\frac{c^\delta}{\delta} + \beta\frac{l^\delta}{\delta} - \lambda(c + wl - wT - Y)$$

and the FOCs are

$$\alpha c^{\delta-1} - \lambda = 0$$
$$\beta l^{\delta-1} - w\lambda = 0$$

so rearranging, we find that

$$c\left(\frac{\alpha w}{\beta}\right)^{\frac{1}{\delta-1}} = l$$

and plugging this into the budget constraint $c = w(T - l) + Y$, we find the demand function for $c$ and $l$.

$$c^*(w, Y) = \frac{wT + Y}{1 + w\left(\frac{\alpha w}{\beta}\right)^{\frac{1}{\delta-1}}}$$

$$l^*(w, Y) = \frac{wT + Y}{w + \left(\frac{\alpha w}{\beta}\right)^{\frac{1}{\delta-1}}}$$

From the demand functions, it is clear that $\frac{\partial c*}{\partial Y} > 0$ and $\frac{\partial l*}{\partial Y} > 0$ for all $Y$ and therefore both consumption and leisure are normal goods.

**d)** How do $c$ and $l$ change as the wage $w$ increases? Show that your result can be interpreted as income and substitution effects. Note: An intuitive answer will get you most of the points.

**Solution**: Intuition: For consumption, both income and substitution effects go in the same direction where higher $w$ leads to higher $c$. For leisure, the income effect leads to higher $l$ but this is counteracted by the substitution effect where higher $w$ leads the worker to want to work more and consume less leisure as $l$ costs $w$. Whether the income effect or substitution effect dominates depends on the exact values.

Algebra: We can rewrite the maximization problem to be in only a single variable since the budget constraint binds.

$$\max U(l) = \alpha\frac{(wT - wl + Y)^{\delta}}{\delta} + \beta\frac{l^{\delta}}{\delta}$$

The FOC is then

$$-w\alpha(wT - wl + Y)^{\delta-1} + \beta l^{\delta-1} = 0$$

Let the RHS be $f(w, Y, l)$. To find the comparative statics of the model, we can totally differentiate the FOC and get

$$\frac{\partial f}{\partial w}dw + \frac{\partial f}{\partial Y}dY + \frac{\partial f}{\partial l}dl = 0 \qquad (*)$$

The partial derivatives are

$$\frac{\partial f}{\partial w} = -\alpha(wT - wl + Y)^{\delta-1} - w\alpha(\delta-1)(wT - wl + Y)^{\delta-2}(T-l)$$
$$\frac{\partial f}{\partial Y} = -w\alpha(\delta-1)(wT - wl + Y)^{\delta-2}$$
$$\frac{\partial f}{\partial l} = w^2\alpha(\delta-1)(wT - wl + Y)^{\delta-2} + \beta(\delta-1)l^{\delta-2}$$

To find what $\frac{dl}{dw}$ is, conditional on $dY = 0$ (so only wages are changing), we can rearrange the equation $(*)$ and plug in partial derivatives to get

$$\begin{aligned}\frac{dl}{dw}|_{dY=0} = \frac{\frac{\partial f}{\partial w}}{\frac{\partial f}{\partial l}} &= \frac{-\alpha(wT - wl + Y)^{\delta-1} - w\alpha(\delta-1)(wT - wl + Y)^{\delta-2}(T-l)}{\frac{\partial f}{\partial l}} \\ &= \frac{-\alpha(wT - wl + Y)^{\delta-1}}{\frac{\partial f}{\partial l}} + (T-l)\frac{-w\alpha(\delta-1)(wT - wl + Y)^{\delta-2}}{\frac{\partial f}{\partial l}} \\ &= \frac{-\alpha(wT - wl + Y)^{\delta-1}}{\frac{\partial f}{\partial l}} + (T-l)\frac{\frac{\partial f}{\partial Y}}{\frac{\partial f}{\partial l}}\end{aligned}$$

And since $\frac{\frac{\partial f}{\partial Y}}{\frac{\partial f}{\partial l}} = \frac{dl}{dY}|_{dw=0}$, this is the income effect. The above derivation is equivalent to the Slutsky Equation where the first term is the substitution effect and the second term is the income effect. From this we see that the substitution effect is negative and the income effect is positive (recall $\delta < 1$).

We can follow the same steps for $c$. The FOC in only $c$ is

$$\alpha c^{\delta-1} - \frac{\beta}{w}(T - \frac{c}{w} + \frac{Y}{w})^{\delta-1} = 0$$

Let the RHS be $g(w, Y, c)$, and totally differentiating, we find that

$$\frac{\partial g}{\partial w}dw + \frac{\partial g}{\partial Y}dY + \frac{\partial g}{\partial c}dc = 0 \qquad (**)$$

The partial derivatives are

$$\frac{\partial g}{\partial w} = \beta w^{-2}(T - \frac{c}{w} + \frac{Y}{w})^{\delta-1} + \frac{\beta}{w^3}(\delta-1)(T - \frac{c}{w} + \frac{Y}{w})^{\delta-2}(-c+Y)$$
$$\frac{\partial g}{\partial Y} = -\frac{\beta}{w^2}(\delta-1)(T - \frac{c}{w} + \frac{Y}{w})^{\delta-2}$$
$$\frac{\partial g}{\partial c} = \alpha c^{\delta-2}(\delta-1) + \frac{\beta}{w^2}(\delta-1)(T - \frac{c}{w} + \frac{Y}{w})^{\delta-2}$$

Then rearranging and plugging partial derivatives, we get

$$\frac{dc}{dw}|_{dY=0} = \frac{\frac{\partial g}{\partial w}}{\frac{\partial g}{\partial c}} = \frac{\beta w^{-2}(T - \frac{c}{w} + \frac{Y}{w})^{\delta-1} + \frac{\beta}{w^3}(\delta-1)(T - \frac{c}{w} + \frac{Y}{w})^{\delta-2}(-c+Y)}{\frac{\partial g}{\partial c}}$$

We can see here that all terms are positive, so the income and substitution effect go in the same direction.

# 3 Problem 3: Production Functions and Feasible Allocations

Recall the Leontief input-output model from lecture 4. Suppose we have two commodities and input-output matrix given by

$$A = \begin{bmatrix} .2 & .7 \\ .6 & .1 \end{bmatrix}$$

Specifically, producing one unit commodity 1 costs .2 units of commodity 1 and .6 units of commodity 2, and producing one unit commodity 2 costs .7 units of commodity 1 and .1 units of commodity 2.

**a)** Suppose John has a demand vector given by $D = [3, 1]$. Find the production vector $X = [X_1, X_2]'$ that satisfies this demand.

**Solution**: We can use the formula from the lecture notes: $X = (I - A)^{-1}D$. Plugging in values, this gives us

$$X = \begin{bmatrix} 11.333 \\ 8.667 \end{bmatrix}$$

**b)** Now suppose John has a utility function given by $U_J(Y_1, Y_2) = \alpha Y_1 + \beta Y_2$ where $\alpha, \beta > 0$. Characterize the set of production vectors $X$ that gives John a utility of $V > 0$. (Hint: this will be a linear equation of $X_1$ and $X_2$ in terms of $\alpha, \beta$, and $V$)

**Solution**: Let $\theta = [\alpha, \beta]$ and $Y = [Y_1, Y_2]'$. Recall that output available for consumption is given by $(I - A)X$, thus we have that $Y = X - AX = (I - A)X$. Thus, the equation that characterizes the set of production vectors $X$ that gives John a utility of $V$ is given by

$$\theta(I - A)X = V$$

writing this out gives us

$$X_1 = \frac{V}{0.8\alpha - 0.6\beta} + \frac{-0.7\alpha + 0.9\beta}{0.8\alpha - 0.6\beta} X_2 \tag{1}$$

**c)** Suppose Sally does not like it when $X_2$ is produced in either too much or too little quantity. Specifically, Sally's utility is given by $U_S(X_2) = -\gamma|X_2 - \overline{X}|$ where $\gamma > 0$. Find the production vector $X^*$ that maximizes Sally's utility subject to keeping John's utility constant at $V$. (Hint: you should not use any calculus to solve this problem)

Sally's utility is maximized at $X_2^* = \overline{X}$. To keep John's utility constant, then we must have $X_1^* = \frac{V}{0.8\alpha - 0.6\beta} + \frac{-0.7\alpha + 0.9\beta}{0.8\alpha - 0.6\beta}\overline{X}$

# 4 Problem 4: Giffen Good

Results are already in the assignment document.