Fall 2018 14.01 Problem Set 3 - *Solutions*

Problem 1 (28 points)

True/False/Uncertain. Please fully explain your answer, including diagrams where appropriate. Points are awarded based on explanations.

1. (4 points) The short run marginal cost curve is always increasing due to the law of diminishing marginal returns.

   Solution: False. The law of diminishing marginal returns means that the marginal cost curve will eventually be increasing. However, at low levels of the input, there can be increasing marginal returns to an input, and MC may decrease at first.

2. (4 points) In the short run, perfectly competitive firms with higher fixed costs must also charge a higher price, all else equal.

   Solution: False. Perfectly competitive firms cannot alter their price - they are price takers of the market price. Furthermore, in the short run, fixed costs are equivalent to sunk costs, and will not affect the firm's decision making. $MC(q) = p$ *will hold, and this condition will determine their level of output.*

3. (4 points) If marginal cost is increasing, then average cost must increase as well.

   Solution: False. Average cost increases when marginal cost is greater than average cost. So marginal cost can be increasing, but still be less than average cost.

4. (4 points) If a firm's production function exhibits decreasing marginal returns in each factor, then it must also exhibit decreasing returns to scale.

   Solution: False. For example, $F(K, L) = K^{3/4}L^{3/4}$.

5. (4 points) Two firms in the same perfectly competitive market, A and B, have short run costs given by $C_A(q) = 10 + 2q^2$ and $C_B(q) = 10 + 3q^2$ respectively. Since B has higher costs, it must charge a higher price **in equilibrium**.

   Solution: False. Perfectly competitive firms cannot alter their price - they are price takers of the market price. In general, firm B *will either sell less at the same price* p *or leave the market.*

6. (4 points) If a firm has U-shaped (convex) marginal cost, then AVC and MC are equal at the point where AVC is minimized.

   Solution: True.

   - *Long run: Note that in the long run $C(q) = VC(q)$.*
     Then, let $C(q)$ be the firm's variable (and total) cost in the long run, so $AVC = \frac{C(q)}{q}$ and $MC = C'(q)$. The derivative of the AVC equals $\frac{C'(q)q - C(q)}{q^2}$, which is 0 if and only if the numerator is 0, i.e., if and only if $AVC = MC$. Finally, we need to argue that this critical point corresponds to a minimum. Since MC is U-shaped, we know that $C''(q) > 0$. Then we can calculate the second derivative of AVC, which is

     $$\frac{C''(q)q^3 - 2(C'(q)q - C(q))q}{q^4} > 0$$

     at the point where $AVC = MC$, since $C''(q) > 0$ and $C'(q)q = C(q)$. Intuitively, we can also argue this graphically: to the left of the intersection, $AVC > MC$ and so AVC is decreasing, while to the right $MC > AVC$ and so AVC is increasing.
   - *Short run: $C(q) = VC(q) + FC \Rightarrow AVC(q) = ATC(q) - \frac{FC}{q} \Rightarrow AVC'(q) = \frac{qC'(q) - C(q) + FC}{q^2}$. To be at a minimum, we need $AVC'(q^*) = 0 \Rightarrow q^*C'(q^*) - C(q^*) + FC = 0 \Rightarrow C'(q^*) = \frac{C(q^*) - FC}{q^*} = AVC(q^*)$.*

   NOTE: we are grading fully both true or false since it was easier to show for when ATC=AVC.

7. *(4 points) A firm with production function $q = L + 2K$ will always hire either labor or capital, but never both at the same time.*

   Solution: False. If $r = 2w$, then the firm is indifferent and may choose any combination of labor and capital.

Problem 2 (16 points)

A firm has a production function $q = f(K, L) = K + 0.5L$ and face wages w and rental rate of capital r. Let $w = 1, r = 1$. For this problem, think about the long-run where capital is not fixed.

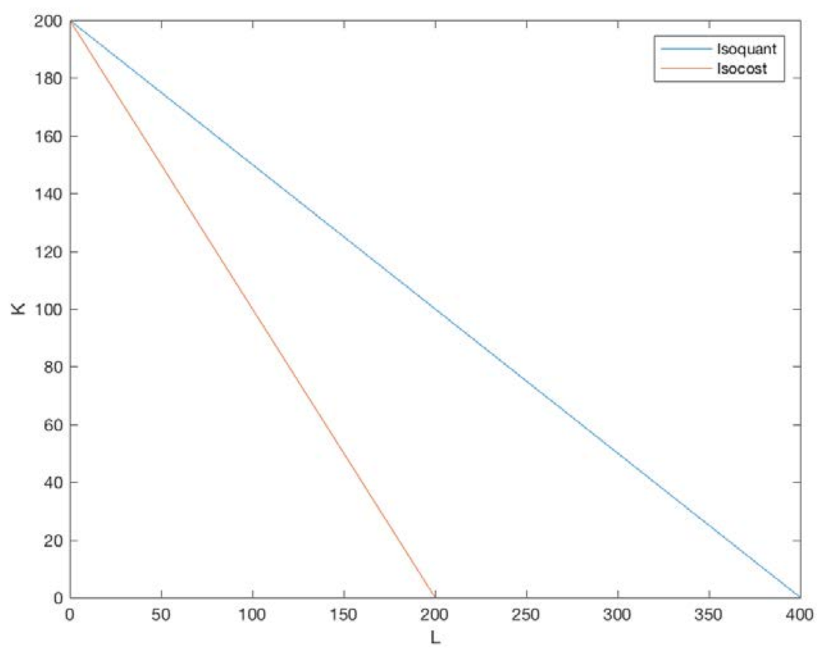
1. (8 points) Suppose the firm wants to produce $q = 200$. What is the combination of K and L that minimizes total cost? Draw the isoquant and isocost curves that correspond to the firm's optimal choice, with K in the y-axis and L in the x-axis. Explain.

   Solution: Note that the two inputs are perfect substitutes in the production function. Since labor and capital cost the same but capital is twice as productive as *labor, the firm will only use capital. Another way to see this is by looking at the MRTS (given by the isoquant curve) and the MRT (given by the isocost curve):*

$$MRTS = \frac{MP_L}{MP_K} = 0.5 \quad \textit{and} \quad MRT = \frac{w}{r} = 1$$

Since both quantities are constant, at any level of K and L it follows that $MRTS < MRT$, which means that labor is relatively more expensive than capital given their relative productivity. Thus, the firm only wants to use capital.
The isoquant curve is given by $K = 200 - 0.5L$. The isocost curve is given by $K + L = 200$, since the minimizing cost is 200 –the firm uses 200 units of capital to produce 200 units of output and the price of capital is 1. The slope of the isocost curve is given by $-w/r = -1$.
Figure 3 plots the corresponding isoquant and isocost curves.

Figure 1: Isoquant and isocost curves



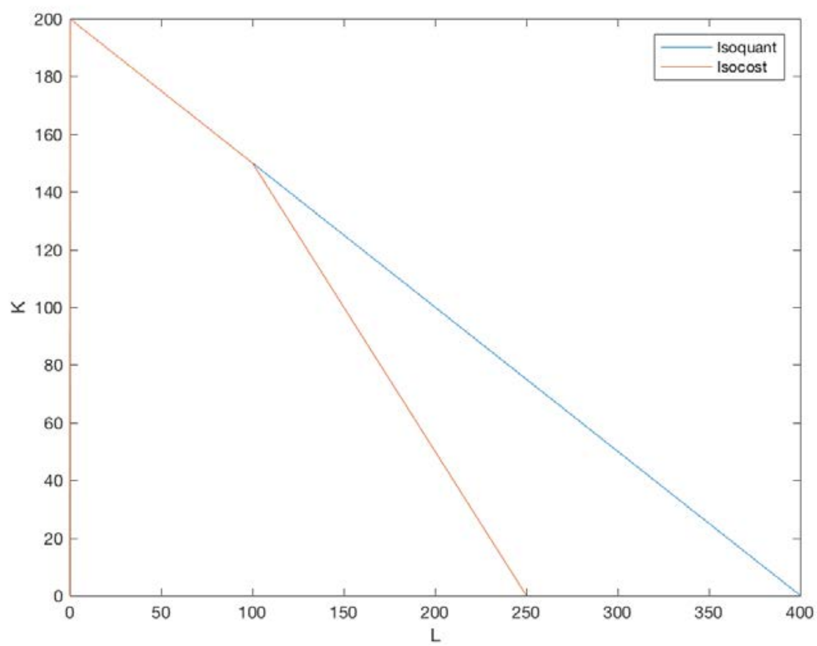
2. (8 points) Suppose that the government wants to encourage the use of labor and decides to pay for 50% of the firm's wage costs for the first 100 units of labor used –i.e. the firm only pays $0.5w$ for the first 100 units of labor used. Again, assume that the firm wants to produce $q = 200$. What are all the possible combinations of K and L that minimize total cost? Draw the isoquant and isocost curves that correspond to the firm's optimal choice. Explain.

   Solution: The isoquant curve does not change –note that the production constraint is still the same.

- *When $L \leq 100$, the MRT becomes $MRT = (0.5w)/r = 0.5$. which implies that $MRTS = MRT$ when $L \leq 100$. That is, the relative productivity of the two inputs equals its relative cost and the firm substitutes indifferently between labor and capital. E.g. the firm will set $L^* \in [0, 100]$ and capital will be given by the production constraint, $K^* = 200 - 0.5L^* \in [150, 200]$.*
- *When $L > 100$, $MRTS < MRT$ again because $MRT = 1$, which implies that the firm will never set $L^* > 100$.*

Therefore, the isocost curve has a slope of -0.5 when $L \leq 100$ but a slope of -1 when $L > 100$, with a kink point when $L = 100$.
Figure 4 plots the corresponding isoquant and the isocost curves.

Figure 2: Isoquant and isocost curves after the policy change

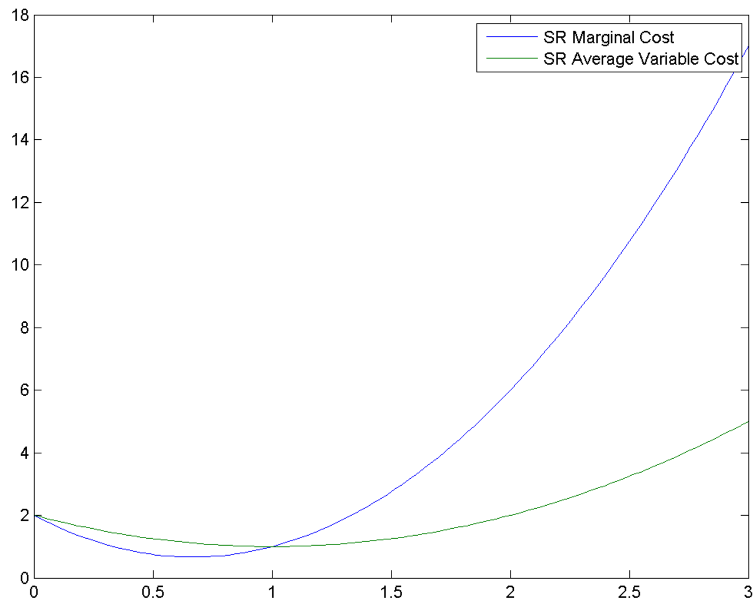


Problem 3 (16 points)

In the short run, a firm has fixed capital $\bar{K}$. We know that its short-run cost function is $C^{SR}(q) = q^3 - 2q^2 + 2q + 2$.

1. (8 points) Plot the short-run marginal cost and average variable cost (as a function of q). What is the short-run supply curve?

Figure 3:



Solution: The short-run supply curve coincides with the marginal cost curve when this is above the average variable cost. We have that

$$\begin{aligned} MC^{SR}(q) &= 3q^2 - 4q + 2 \\ AVC^{SR}(q) &= q^2 - 2q + 2 \end{aligned}$$

We can find the intersection between this two curves:

$$\begin{aligned} 3q^2 - 4q + 2 &= q^2 - 2q + 2 \\ &\Rightarrow q = 1 \end{aligned}$$

Therefore, the supply curve is

$$Supply(p) = \begin{cases} 0 & if\ p < 1 \\ \frac{2+\sqrt{-2+3p}}{3} & if\ p \geq 1 \end{cases}$$

2. (8 points) Suppose that the long-run cost curve is $C^{LR}(q) = \frac{3}{2}q^2$. Can you find the quantity $\bar{q}$ such that in the long-run the firm optimally chooses to use $\bar{K}$ units of capital to produce $\bar{q}$? (*Hint: if you are having trouble finding the solution by hand, use a numerical solver; Wolfram Alpha is a great resource!*)
Solution: $\bar{q}$ will be such that the long-run and the short-run costs coincide:

$$q^3 - 2q^2 + 2q + 2 = \frac{3}{2}q^2$$

So we get that $\bar{q} = 2$.

Problem 4 (40 points)

(This is a somewhat mathematically involved problem. Please show your work, partial credit will be given)

A firm has a Cobb-Douglas production function $q = f(K, L) = K^{\alpha}L^{1-\alpha}$ and faces wages, w, and rental rate of capital, r.

1. (3 points) Does this production function exhibit increasing, decreasing, or constant returns to scale?

   Solution: Constant returns to scale. $f(\lambda L, \lambda K) = (\lambda K)^{\alpha}(\lambda L)^{1-\alpha} = \lambda K^{\alpha}L^{1-\alpha} = \lambda f(K, L)$

2. (6 points) Find the *short-run* cost curve, $C(q)$, as a function of q and the parameters.

   Solution: Plugging in $\bar{K}$ in the production function and rearrange the production function we get:

   $$q = \bar{K}^{\alpha}L^{1-\alpha} \Rightarrow L^*(q) = \left(\frac{q}{\bar{K}^{\alpha}}\right)^{\frac{1}{1-\alpha}}$$

   The total cost function is given by $C = rK + wL$, so

   $$C(q) = r\bar{K} + wL^*(q) = r\bar{K} + w\left(\frac{q}{\bar{K}^{\alpha}}\right)^{\frac{1}{1-\alpha}}$$

3. (6 points) **For this subpart**, assume that $\bar{K} = 10$, $r = 1.5$, $w = 6$, and $\alpha = 2/3$. Derive expressions for MC, VC, FC, ATC, AVC, and AFC. Plot MC, ATC, AVC, and AFC, all on the same graph (using a graphing program –WolframAlpha, Mathematica, Matlab, etc.– is fine for this part).

   Solution:

   $$C(q) = 10 \times 1.5 + 6\left(\frac{q}{10^{2/3}}\right)^{1/(1-2/3)} = 15 + \frac{6}{100}q^3$$
   $$VC(q) = \frac{6}{100}q^3$$
   $$FC = 15$$

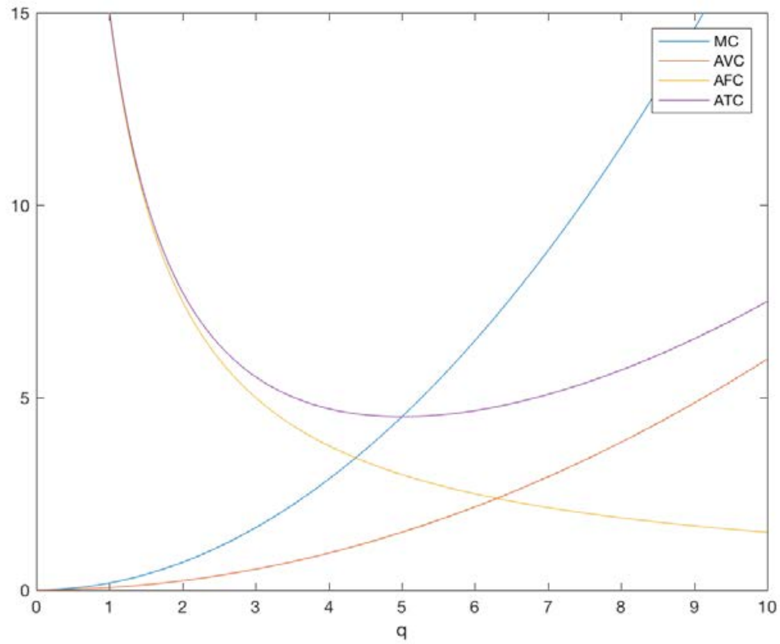
   MC, ATC, AVC and AFC are given by:

   $$MC = \frac{18}{100}q^2 = \frac{9}{50}q^2,\ ATC = \frac{C(q)}{q} = 0.06q^2$$

   $$AVC = \frac{VC(q)}{q} = 0.06q^2,\ AFC = \frac{FC}{q} = \frac{15}{q}$$

   These 4 expressions are plotted on Figure 4.

Figure 4: MC, ATC, AVC, and AFC



4. (6 points) **For this subpart**, assume that $\bar{K} = 10$, $r = 1.5$, $w = 6$, and $\alpha = 2/3$. Assume now that we know the market price is $p = 18$, which is fixed, and we are still operating in the short-run. What is the profit-maximizing choice of q?

   Solution:

$$\pi = R - C = pq - C(q)$$

Taking the derivative with respect to q and setting to zero, we get the standard condition that $p = MC$. From $C(q)$ above, we can write:

$$MC = \frac{1}{1-\alpha} w \left(\frac{q}{\bar{K}}\right)^{\frac{\alpha}{1-\alpha}} = p$$

$$q^* = \left[(1-\alpha)\frac{p}{w}\right]^{\frac{1-\alpha}{\alpha}} \bar{K}$$

Substituting the values,

$$q^* = \left[\left(1 - \frac{2}{3}\right)\frac{18}{6}\right]^{\frac{1-\frac{2}{3}}{\frac{2}{3}}} 10 = \sqrt{1} \cdot 10 = 10$$

5. (6 points) Solve for profits, π, as a function of market price, p (and the parameters w, r, $\bar{K}$, α). Then, assume as we did in subpart 3 that $\bar{K} = 10$, $r = 1.5$, $w = 6$, and $\alpha = 2/3$, and continue to assume so until subpart 6 (included). Will profits ever be negative? If so, find the price range at which profits are negative.

   Solution:

$$\pi(p) = pq^* - C(q^*) = p\left[(1-\alpha)\frac{p}{w}\right]^{\frac{1-\alpha}{\alpha}}\bar{K} - w\left(\frac{\left[\left[(1-\alpha)\frac{p}{w}\right]^{\frac{1-\alpha}{\alpha}}\bar{K}\right]}{\bar{K}^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} - r\bar{K}$$

$$\pi(p) = \alpha\bar{K}p\left(\frac{(1-\alpha)p}{w}\right)^{\frac{1-\alpha}{\alpha}} - r\bar{K} \tag{1}$$

   Profits will therefore be negative when:

$$\alpha\bar{K}p\left(\frac{(1-\alpha)p}{w}\right)^{\frac{1-\alpha}{\alpha}} < r\bar{K}$$

$$p < \left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}$$

   So the price range as a function of the parameters at which profits are negative is

$$0 \leq p < \left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha} \tag{2}$$

   Plugging in the parameter values, the price range where profits are negative is

$$0 \leq p < \left(\frac{1.5}{2/3}\right)^{2/3}\left(\frac{6}{1-(2/3)}\right)^{1-(2/3)}$$

$$0 \leq p < 4.5 \tag{3}$$

   Note: *Full credit for the correct solution for $\pi(p)$ as a function of the parameters, as given in (1), solving for the price range in (3), and noting profits will be negative in this range.*

6. (6 points) Is the firm better off shutting down (producing $q = 0$) at any positive low price? Explain.

Solution: False. At $q = 0$, profits are still negative:

$$\pi = pq - C(q) = 0p - w\left(\frac{0}{\bar{K}^{\alpha}}\right)^{\frac{1}{1-\alpha}} - r\bar{K} = -r\bar{K} = -15$$

This is strictly less than producing the profit-maximizing choice of q, even if the resulting profits from doing so are also negative:

$$\pi(p) = \alpha\bar{K}p\left((1-\alpha)\frac{p}{w}\right)^{\frac{1-\alpha}{\alpha}} - r\bar{K}$$

$$= (2/3)10p\left(\frac{p(1/3)}{6}\right)^{\frac{1/3}{2/3}} - 15$$

$$\pi(p) = \frac{20}{9\sqrt{2}}p^{\frac{3}{2}} - 15$$

At any positive price p, producing will be the better decision. Therefore the firm will always produce.

You may alternatively show this answer by using the condition discussed in class for whether a firm will shut down short-term:

$$p < AVC(q^*)$$

$$AVC(q^*) = \frac{w\left(\frac{q^*}{\bar{K}^{\alpha}}\right)^{\frac{1}{1-\alpha}}}{q^*} = w\left(\frac{q^*}{\bar{K}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$q^* = \left((1-\alpha)\frac{p}{w}\right)^{\frac{1-\alpha}{\alpha}}\bar{K}$$

$$AVC = w\left(\left((1-\alpha)\frac{p}{w}\right)^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\alpha}{1-\alpha}} = p(1-\alpha)$$

Given $0 < \alpha < 1$, the condition $p\alpha > p$ will never hold, and the firm will always produce.

7. (7 points) Let's think about the firm's problem in the long run. Find their optimal choices of inputs (as a function of q, w, r, α) and the resulting long-run cost function $C(q)$. How does the firm's choice of L and K change if r goes up? What happens to costs?

   Solution: We find the optimal choice of Labor and Capital by equating MRTS and -(w/r)

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{w}{r}$$

$$\frac{(1-\alpha)(\frac{K}{L})^{\alpha}}{\alpha(\frac{L}{K})^{1-\alpha}} = \frac{w}{r}$$

$$\frac{1-\alpha}{\alpha}\frac{K}{L} = \frac{w}{r}$$

   Plugging into the production function,

$$L^*(q) = q\left(\frac{r}{w}\frac{1-\alpha}{\alpha}\right)^{\alpha}$$

$$K^*(q) = q\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$$

$$C(q) = wL^*(q) + rK^*(q) = qw\left(\frac{r}{w}\frac{1-\alpha}{\alpha}\right)^{\alpha} + qr\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$$

$$C(q) = qr^{\alpha}w^{1-\alpha}\left[\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\right]$$

   If r goes up, then L^* will increase, K^* will decrease, and total costs will increase, which is all intuitive.