Problem Set 8

The Lorenz model is given by

$$\begin{aligned}\dot{X} &= \mathrm{Pr}Y - \mathrm{Pr}X \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ\end{aligned}$$

where Pr is the Prandtl number (usually taken to equal 10), b is a constant (usually taken to equal 8/3) and r is the ratio of the Rayleigh number to the critical Rayleigh number.

1. Find the location (analytically) of the three steady state solutions for the Lorenz model. Give a physical interpretation of these steady state solutions.

2. Show analytically that the solution corresponding to conduction becomes unstable for $r > 1$.

3. Show analytically that the steady-state solutions corresponding to convection become unstable for
$$r > r_c = \frac{\mathrm{Pr}(\mathrm{Pr} + 3 + b)}{\mathrm{Pr} - 1 - b}$$

4. Use `lorenz` to verify numerically (with plots) that steady-state convection is unstable for $r > r_c$ (use Pr=10, b=8/3). $r = 28$ gives nice results. Give a physical interpretation of the time dependent behaviour you have found for $X(t)$, $Y(t)$, and $Z(t)$. You may wish to make Poincaré sections and/or projections in the XY, XZ, and YZ planes.

5. Find $X(t)$ for $r = 166$ and $r = 166.1$. What observations can you make? Remember to let the transients die out.

6. Verify exponential divergence of small differences in initial conditions for $r = 170$. Assuming $\delta(t) = \delta_0 e^{\lambda t}$, where $\delta(t)$ is the distance between two points in phase space, find the value of λ, the largest Lyapanov exponent. What accounts for the long-time behaviour of $\delta(t)$?

7. Obtain another example of your own choosing of some interesting behaviour of the Lorenz model and describe what you found.