Problem Set 7

1. In this problem, you will derive the linear stability relations for a special case of Rayleigh-Bénard convection.

   The equations describing thermal convection are

$$\begin{aligned} \frac{1}{\mathrm{Pr}}\left(\frac{\partial \vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right) &= -\vec{\nabla}p+\theta\hat{z}+\nabla^2\vec{v} &(1)\\ \frac{\partial\theta}{\partial t}+\vec{v}\cdot\vec{\nabla}\theta &= \nabla^2\theta+\mathrm{Ra}\ w &(2)\\ \vec{\nabla}\cdot\vec{v} &= 0 &(3)\end{aligned}$$

   where $\hat{z}$ is the vertical unit vector, $\vec{v}$ the velocity, w the vertical velocity, Pr the Prandtl number, Ra the Rayleigh number, and θ the deviation of the temperature from the value it would have in the absense of convection. By taking the curl of the first equation twice and neglecting nonlinear terms from both, we obtain the following set of linear equations for the vertical velocity valid for small values of w and θ:

$$\begin{aligned} \frac{1}{\mathrm{Pr}}\nabla^2\left(\frac{\partial w}{\partial t}\right) &= \nabla_1^2\theta+\nabla^4 w &(4)\\ \frac{\partial\theta}{\partial t} &= \nabla^2\theta+\mathrm{Ra}\ w. &(5)\end{aligned}$$

   Here $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, the horizontal Laplacian $\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, and $\nabla^4=(\nabla^2)^2$. The fluid is confined between the planes $z=0$ and $z=1$, so $w=0$ and $\theta=0$ at these boundaries. Convection experiments usually have rigid boundaries, but for mathematical simplicity we consider free boundaries, at which tangential stress vanishes. This yields the additional boundary condition $\frac{\partial^2 w}{\partial z^2}=0$ at $z=0$ and $z=1$. To derive the conditions under which a purely conductive system is unstable, we consider small sinusoidal perturbations of the velocity and temperature fields:

$$\begin{aligned} w &= A(t)\cos(k_x x+k_y y)\sin\pi z &(6)\\ \theta &= B(t)\cos(k_x x+k_y y)\sin\pi z. &(7)\end{aligned}$$

   Here k_x and k_y are the horizontal wavenumbers (ie. $k_x=2\pi/\lambda_x$, where λ_x is the wavelength of the perturbation in the x direction).

   (a) Verify that the sinusoidal perturbations given in equations (6) and (7) satisfy the boundary conditions.

   (b) Substitute (6) and (7) into (4) and (5) to obtain a set of linear ODEs describing A and B. Determine the fixed point, and show that it is unstable when

$$\mathrm{Ra}>\mathrm{Ra}^*=\frac{(k^2+\pi^2)^3}{k^2},$$

where $k^2 = k_x^2 + k_y^2$. Sketch a graph of $\mathrm{Ra}^*(k)$. What physical state of the system does the fixed point correspond to?

(c) What is the value of the minimum, called Ra_c, of the function $\mathrm{Ra}^*(k)$? At what value of k does it occur, and how does the wavelength corresponding to this critical wavenumber compare to the distance between the free boundaries of the fluid? For comparison, $\mathrm{Ra}_c = 1707$ for rigid-rigid boundaries and 1100 for rigid-free boundaries.

2. Thermal gradients in convecting fluids can be measured by shining beams of light into the fluid. `exp1` and `exp2` are two time series from a single experiment performed by Monique Dubois and Pierre Bergé. Each time series measures a quantity proportional to the thermal gradient ΔT, at locations sufficiently separated in space so that they record distinct signals. The Rayleigh number is well above the critical value for convection. Using power spectra and phase space reconstructions, tell us what you can about the convective regime revealed by the data.

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