Problem Set 4

Reading: Strogatz, sections 8.2 and 8.4

1. In class we discussed the FitzHugh-Nagumo model of neuronal excitability. To make it more analytically tractable, we modify it slightly to read:

$$\dot{x} = -y + x(a - x)(x - 1), \tag{1}$$
$$\dot{y} = bx - cy, \tag{2}$$

where b, c are positive constants.

   (a) What are the nullclines of the system?

   (b) There is a fixed point at (0,0). Show that the eigenvalues of the Jacobian evaluated at this fixed point are

$$\lambda = \frac{-a - c \pm \sqrt{(a+c)^2 - 4(ac+b)}}{2}. \tag{3}$$

   (c) Make sketches of where your λ values lie in the complex plane, for different values of a. You should find three qualitatively different regimes: at what values of a do the transitions occur? What does each regime mean in terms of the fixed point's stability?

   (d) You should just have found a Hopf bifurcation. At what value of a does it occur? We'll call this a_{Hopf}.

   (e) `fitzhugh_nagumo` integrates Eqs. (1) and (2) and plots the nullclines. Use $b = 0.01$, $c = 0.02$, and the initial condition (0.2,0) as provided. Plot trajectories in the phase plane, for different values of a, and find the value of a at which the limit cycle first appears: let's call this a_{cycle}[1]. How close is this to your estimate of the Hopf bifurcation above (a_{Hopf})?

   (f) You should have found that $a_{\mathrm{cycle}} > a_{\mathrm{Hopf}}$. In the small regime between these two values, there exist two stable attractors: a limit cycle and a fixed point. Demonstrate this numerically by using `fitzhugh_nagumo` and making phase plane plots with different initial conditions (Hint: the fixed point is at the intersection of the nullclines).

   (g) Between the stable limit cycle and the stable fixed point, there is an unstable limit cycle. At $a = a_{\mathrm{cycle}}$ it collides with the stable limit cycle in a *saddle-node bifurcation of cycles*, and at $a = a_{\mathrm{Hopf}}$ it collides with the stable fixed point. We could find this cycle by integrating backwards in time, but we won't do so here.

   Knowing all of this, you should be able to categorize the Hopf bifurcation you found as supercritical or subcritical: which is it?

[1] Don't get confused by the system taking longer and longer to get to the stable fixed point: the limit cycle should be large and unambiguous.

(h) Set $a = 0.02$. This is a regime in which there is only one stable fixed point, but the system is *excitable*: small perturbations can lead to large one-off disruptions. Demonstrate this numerically by setting the initial condition to be $(x_0, 0)$, and trying different values of x_0. Find one value of x_0 for which the system relaxes quickly back to the fixed point, and one for which the system undergoes an excitation.

(i) Identify, to the best of your ability, the *threshold value* of x_0 for triggering excitations. As we discussed in class, the transition is not discontinuous: it is completely continuous, but there is a small regime in which the response changes extremely rapidly in terms of x_0.