Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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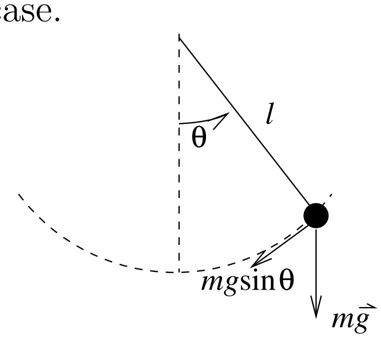
Contents

1 Pendulum

1.1 Free oscillator

The archetypal oscillatory dynamical system is the pendulum. We begin with the unforced, undamped case.



The arc length (displacement) between the pendulum's current position and rest position ($\theta = 0$) is

$$s = l\theta$$

Therefore

$$\begin{aligned}\dot{s} &= l\dot{\theta} \\ \ddot{s} &= l\ddot{\theta}\end{aligned}$$

From Newton's 2nd law,

$$F = ml\ddot{\theta}$$

The restoring force is given by $-mg\sin\theta$. (It acts in the direction opposite to $\mathrm{sgn}(\theta)$). Thus

$$F = ml\ddot{\theta} = -mg\sin\theta$$

or

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0.$$

Our pendulum equation is idealized: it assumes, e.g., a point mass, a rigid geometry, and most importantly, *no friction.*

The equation is nonlinear, because of the $\sin\theta$ term. Thus the equation is not easily solved.

However for small $\theta \ll 1$ we have $\sin\theta \simeq \theta$. Then

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta$$

whose solution is

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t + \phi\right)$$

or

$$\theta = \theta_0 \cos(\omega t + \phi)$$

where the angular frequency is

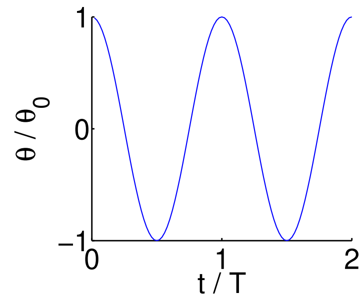
$$\omega = \sqrt{\frac{g}{l}},$$

the period is

$$T = 2\pi\sqrt{\frac{l}{g}},$$

and θ_0 and ϕ come from the initial conditions.

Note that the motion is exactly periodic.
Furthermore, the period T is independent of the amplitude θ_0.



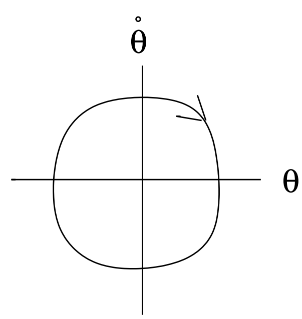
1.2 Global view of dynamics

What do we need to know to completely describe the instantaneous state of the pendulum?

$$\text{The position } \theta \text{ and the velocity } \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}.$$

Instead of integrating our o.d.e. for the pendulum, we seek a representation of the solution in the plane of θ and $\dot{\theta}$.

Because the solution is periodic, we know that the resulting trajectory must be closed:

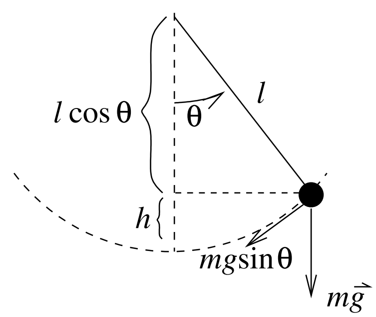


In which direction is the flow?
What shape does the curve take?

To calculate the curve, we note that it should be characterized by constant energy, since no energy is input to the system (it is not driven) and none is dissipated (there is no friction).

Therefore we compute the energy $E(\theta,\dot\theta)$, and expect the trajectories to be curves of $E(\theta,\dot\theta) = \text{const}$.

1.3 Energy in the plane pendulum



The pendulum's height above its rest position is $h = l - l\cos\theta$.
As before, $s = \text{arc length} = l\theta$.

The kinetic energy T is

$$T = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}m(l\dot\theta)^2 = \frac{1}{2}ml^2\dot\theta^2$$

The potential energy U is

$$\begin{aligned} U = mgh &= mg(l - l\cos\theta) \\ &= mgl(1-\cos\theta) \end{aligned}$$

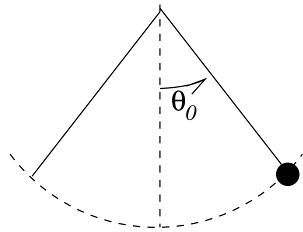
Therefore the energy $E(\theta,\dot\theta)$ is

$$E(\theta,\dot\theta) = \frac{1}{2}ml^2\dot\theta^2 + mgl(1-\cos\theta)$$

We check that $E(\theta,\dot\theta)$ is a constant of motion by calculating its time derivative:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{1}{2}ml^2(2\dot\theta\ddot\theta) + mgl\dot\theta\sin\theta \\ &= ml^2\dot\theta\left(\ddot\theta + \frac{g}{l}\sin\theta\right) \\ &= 0 \qquad \text{(since the pend. eqn. } \ddot\theta = -\frac{g}{l}\sin\theta) \end{aligned}$$

So what do these curves look like?
Take θ_0 to be the highest point of motion.



Then

$$\dot{\theta}(\theta_0) = 0$$

and

$$E(\theta_0, \dot{\theta}\mid_{\theta_0}) = mgl(1 - \cos\theta_0)$$

Since $\cos\theta = 1 - 2\sin^2(\theta/2)$,

$$\begin{aligned} E(\theta_0, \dot{\theta}\mid_{\theta_0}) &= 2mgl\sin^2\left(\frac{\theta_0}{2}\right) \\ &= E(\theta, \dot{\theta}) \text{ in general, since } E \text{ is conserved} \end{aligned}$$

Now write $T = E - U$:

$$\frac{1}{2}ml^2\dot{\theta}^2 = 2mgl\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \qquad (1)$$

$$\dot{\theta}^2 = 4\frac{g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \qquad (2)$$

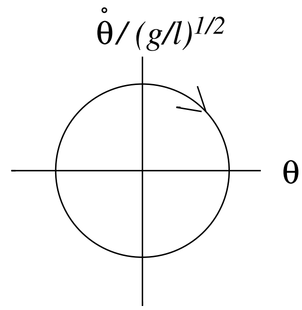
For small θ_0 such that $\theta \ll 1$,

$$\dot{\theta}^2 \simeq 4\frac{g}{l}\left(\frac{\theta_0^2}{4} - \frac{\theta^2}{4}\right)$$

or

$$\left(\frac{\dot{\theta}}{\sqrt{g/l}}\right)^2 + \theta^2 \simeq \theta_0^2$$

Thus for small θ the curves are circles of radius θ_0 in the plane of θ and $\dot{\theta}/\sqrt{g/l}$.



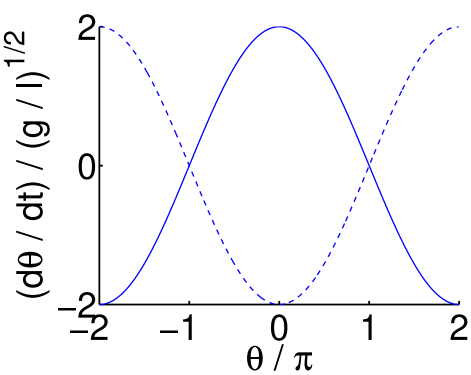
What about θ_0 large?
Consider the case $\theta_0 = \pi$.

For $\theta_0 = \pi$, $E = 2mgl$, and equation (2) gives

$$\begin{aligned}\dot{\theta}^2 &= 4\,\frac{g}{l}\left[\sin^2\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]\\ &= 4\,\frac{g}{l}\cos^2\left(\frac{\theta}{2}\right)\end{aligned}$$

Thus for $\theta_0 = \pi$, the curves are the cosines

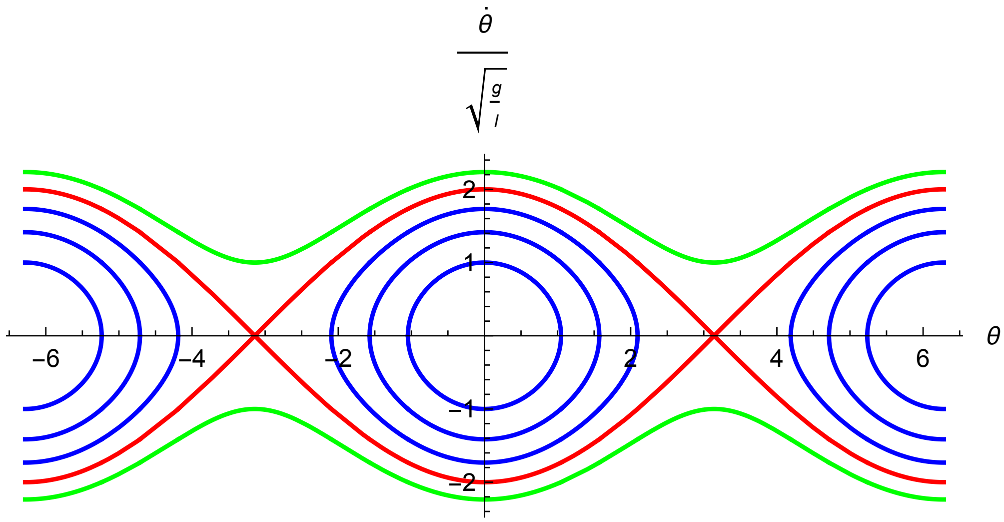
$$\dot{\theta} = \pm 2\sqrt{\frac{g}{l}}\,\cos\left(\frac{\theta}{2}\right). \tag{3}$$



1.4 Phase portrait

Intuitively, we recognize that the cosines of equation (3) separate oscillatory motion ($E < 2mgl$) from rotary motion ($E > 2mgl$).

Thus for undampled, nonlinear pendulum we can construct the following *phase portrait*:



The portrait is periodic.

The blue curves are the oscillatory solutions. They intersect the θ-axis at $\pm\theta_0$, and are circles for small θ_0. The period of motion is independent of θ_0 in the circular case, otherwise the period grows with the amplitude.

The red curves are given by equation (3). The separate oscillatory from rotary motion, which is indicated by the green curve.

In which direction do the closed trajectories flow (counterclockwise...). And the others?

The points $\dot{\theta} = 0$, $\theta = \ldots, -2\pi, 0, 2\pi, \ldots$ are *marginally stable* fixed points.

The points $\dot{\theta} = 0$, $\theta = \ldots, -3\pi, -\pi, \pi, 3\pi \ldots$ are *unstable* fixed points.

The red trajectories appear to cross, but they do not. *Why not?*

If the trajectories actually arrive at these crossing points, the motion stops, awaiting instability. But it takes an infinite time to arrive at these points.

2 Stability in two-dimensional systems

How can we address the question of stability in such two-dimensional systems?

We proceed from the example of the pendulum equation. We reduce this 2nd-order ODE,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0,$$

to two first order ODE's.

Write $x_1 = \theta$, $x_2 = \dot{\theta}$. Then

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{g}{l}\sin x_1 \end{aligned}$$

The fixed points occur where

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \vec{0}$$

For the pendulum, this requires

$$\begin{aligned} x_2 &= 0 \\ x_1 &= \pm n\pi, \quad n = 0, 1, 2, \ldots \end{aligned}$$

Since $\sin x_1$ is periodic, the only distinct fixed points are

$$\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

Intuitively, the first is stable and the second is not.
How may we be more precise?

2.1 Linear systems

Reference: Chapter 5, Strogatz [1].

Consider the problem in general. First, assume that we have the linear system

$$\begin{aligned} \dot{u_1} &= a_{11}u_1 + a_{12}u_2 \\ \dot{u_2} &= a_{21}u_1 + a_{22}u_2 \end{aligned}$$

or

$$\dot{\vec{u}} = A\vec{u}$$

with

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Assume A has an inverse and that its eigenvalues are distinct. Then the only fixed point (where $\dot{\vec{u}} = 0$) is $\vec{u} = 0$.

The solution, in general, is

$$\vec{u}(t) = \alpha_1 e^{\lambda_1 t} \vec{c_1} + \alpha_2 e^{\lambda_2 t} \vec{c_2}$$

where

- λ_1, λ_2 are eigenvalues of A.
- $\vec{c_1}$, $\vec{c_2}$ are eigenvectors of A.
- α_1 and α_2 are constants (deriving from initial conditions).

Recall that the eigenvalues λ of A solve the *characteristic equation*

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$$

which may be written as

$$\lambda^2 - \tau\lambda + \Delta = 0$$

where

$$\begin{aligned} \tau &= \text{trace}(A) = a_{11} + a_{22} \\ \Delta &= \det(A) = a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Then

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}.$$

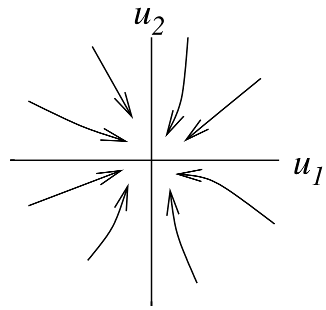
The eigenvectors $\vec{c}$ solve

$$A\vec{c} = \lambda\vec{c}.$$

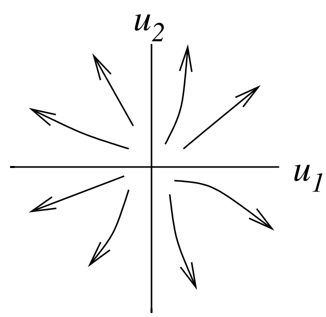
What are the possibilities for stability?

1. λ_1 and λ_2 are both real.

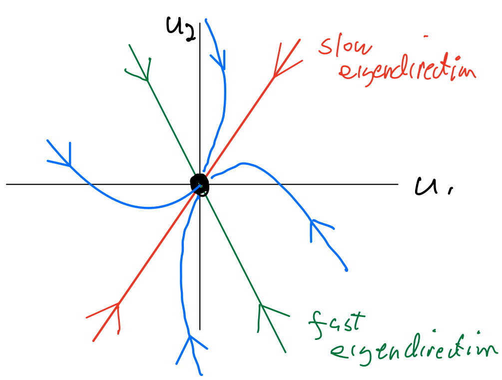
(a) If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $u(t) \to 0$ as $t \to \infty$.
$\Rightarrow$ *stable node.*



(b) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $u(t) \to \infty$ as $t \to \infty$.
$\Rightarrow$ *unstable node.*



Although the above plots make the basic point, they skip over a detail: the difference between fast and slow eigendirections. Here's a more detailed view of a *stable node*, with $\lambda_2 < \lambda_1 < 0$:



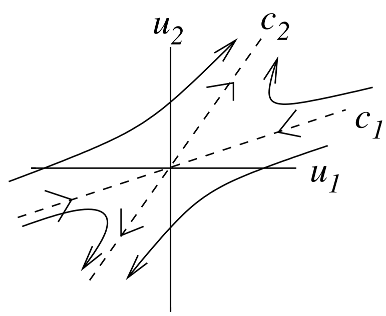
Note that the trajectories eventually become parallel to the slow eigendirection (associated with λ_1).

In the *unstable* case, the arrows are reversed (as if we simply reversed time $t \to -\infty$) and the trajectories become parallel to the fast eigendirection.

(c) If $\lambda_1 < 0 < \lambda_2$,

- If $\vec{u}(0)$ is a multiple of $\vec{c_1}$, then $u(t) \to 0$ as $t \to \infty$.
- If $\vec{u}(0)$ is a multiple of $\vec{c_2}$, then $u(t) \to \infty$ as $t \to \infty$.

$\Rightarrow$ *unstable saddle.*



2. λ_1, λ_2 are both complex. Then

$$\lambda = \sigma \pm iq.$$

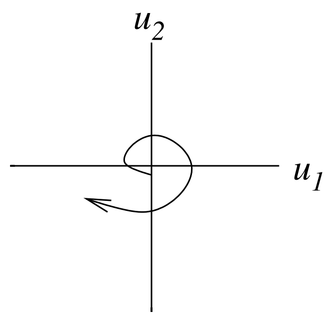
Assuming $\vec{u}(t)$ is real,

$$\vec{u}(t) = e^{\sigma t}(\vec{\beta_1} \cos qt + \vec{\beta_2} \sin qt)$$

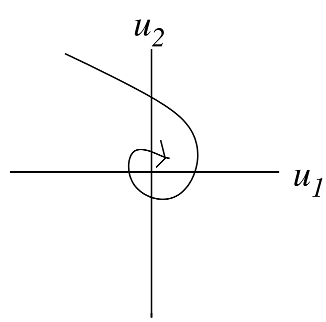
($\vec{\beta_1}, \vec{\beta_2}$ are formed from a linear combination of A's eigenvectors and the initial conditions).

There are three possibilities:

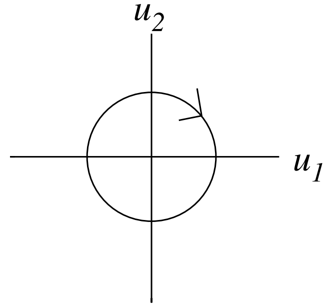
(a) $\mathrm{Re}\{\lambda\} = \sigma > 0 \implies$ *unstable.*



(b) $\sigma < 0 \implies$ *stable.*

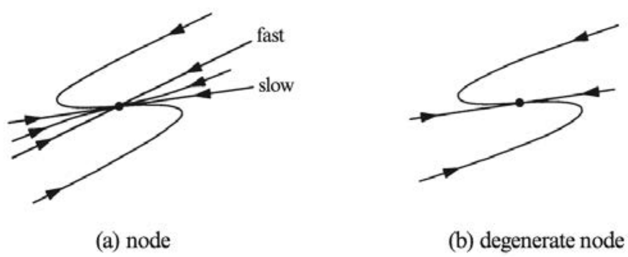


(c) $\sigma = 0 \implies$ *marginally stable*, or a *center*.



The analysis above assumes the usual case of distinct eigenvalues in which $\lambda_1 \neq \lambda_2$.

The case where $\lambda_1 = \lambda_2$ and there is only one eigenvector can be thought of as a limiting case of the fast and slow eigendirections pictured above; it is as if the two eigendirections act as a scissor that closes:



Strogatz [1], Fig. 5.2.7 See image credit at the bottom of Page 14.

Such a node is called *degenerate*. The case above is stable; the unstable case would have the trajectories reversed.

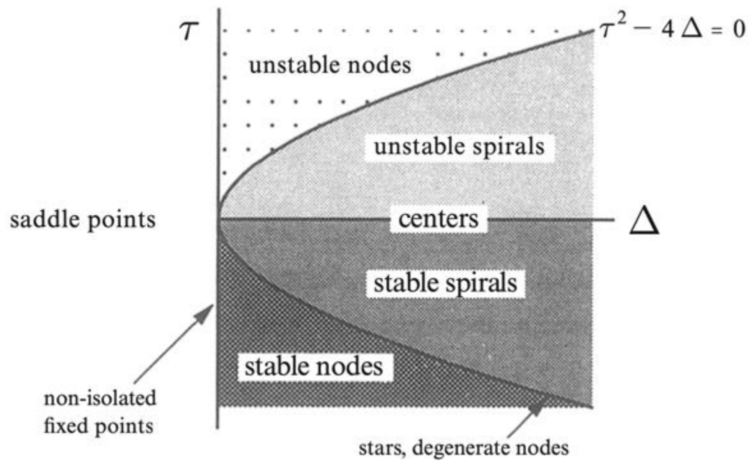
The case with two eigenvectors yields either *stars* (nodes with rays emanating from or converging to the origin) or non-isolated fixed points. See Strogatz [1] for details.

2.2 Classification scheme

All of the cases in the previous section can be conveniently classified in terms of the trace τ and determinant Δ of the matrix A. We have

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2$$

In graphical form, we have



Strogatz [1], Fig. 5.2.8

See image credit at the bottom of Page 14.

2.3 Nonlinear systems

We are interested in the qualitative behavior of systems like

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2) \\ \dot{x_2} &= f_2(x_1, x_2) \end{aligned}$$

where f_1 and f_2 are nonlinear functions of x_1 and x_2.

Suppose $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a fixed point. Is it stable?

Define $u_i = x_i - x_i^*$ to be a small departure from the fixed point.

As we did with one dimensional systems, we expand the system around the fixed point:

$$f_i(x_1, x_2) = f_i(x_1^*, x_2^*) + u_1 \left.\frac{\partial f_i}{\partial x_1}\right|_{x_1^*, x_2^*} + u_2 \left.\frac{\partial f_i}{\partial x_2}\right|_{x_1^*, x_2^*} + O(u^2).$$

The first term vanishes since it is evaluated at the fixed point.

Also, since

$$u_i = x_i - x_i^*$$

we have

$$\dot{u}_i = \dot{x}_i = f_i(x_1, x_2)$$

Substituting $\dot{u}_i = f_i(x_1, x_2)$ above, we obtain

$$\dot{\vec{u}} \simeq A\vec{u}$$

where

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{\vec{x}=\vec{x}^*}$$

A is the *Jacobian* matrix of f at $\vec{x}^*$.

We now apply these results to the pendulum. We have

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) = -\frac{g}{l}\sin x_1 \end{aligned}$$

and

$$A = \begin{pmatrix} 0 & 1 \\ -g/l & 0 \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

There is a different A for the case $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$. (The sign of g/l changes.)

The question of stability is then addressed just as in the linear case, via calculation of the eigenvalues (or the trace and determinant) of the Jacobian.

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).

Image Credit

12.006J/18.353J/2.050J Nonlinear Dynamics: Chaos
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