Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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Contents

1 Flows in one dimension

Reference: Strogatz, Chapter 2 [1].

The simplest dynamical systems concern the evolution of only one variable. Here we consider *flows* in which the the dynamics

$$\dot{x} = f(x)$$

describe the continuous evolution of $x(t)$.

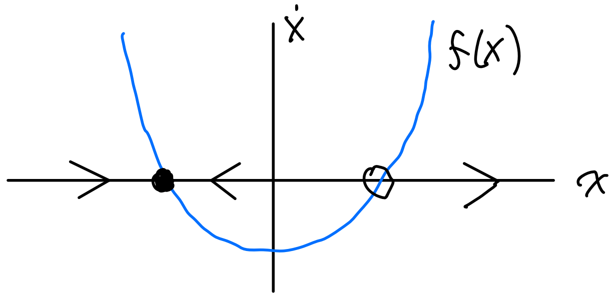
1.1 Fixed points and stability

1.1.1 General statements

Suppose, for example, that

$$f(x) = x^2 - 1.$$

Since a graph of $f(x)$ is equivalent to a graph of $\dot{x}$ as a function of x, we have



The sign of $f(x)$ determines the directions of the *flow* of x; we signify this by arrows.

$f(x)$ intersects the x-axis when $f(x) = 0$. These are called *fixed points* (or equilibrium solutions) of the system. We shall denote these by x^*. Here we have

$$x^* = \pm 1.$$

A fixed point is *stable* if flows are attracted to it.

A fixed point is *unstable* if flows are repelled from it.

By inspection we see that

$$\begin{aligned} f'(x^*) &< 0, \qquad x^* \text{ stable} \\ f'(x^*) &> 0, \qquad x^* \text{ unstable} \end{aligned}$$

1.1.2 Example: logistic growth

Consider the growth of some population, say, rabbits, bacteria, or people. Let

$$\begin{aligned} N &= \text{population size} \\ r &= \text{growth rate.} \end{aligned}$$

The simplest possible growth model has N growing at rate $r > 0$:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN, \qquad r > 0.$$

The solution predicts *exponential growth*:

$$N(t) = N(0)e^{rt}.$$

Such a model works works well when the resource that a population requires for growth is sufficiently abundant.

In cases of such unlimited growth,

$$r = \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = \text{'per capita' or 'specific' growth rate}$$

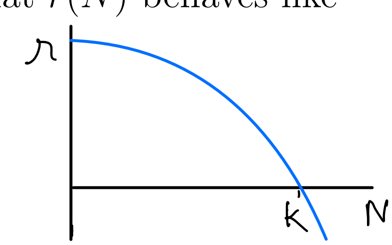
is constant, independent of N.

Eventually, however, resources become depleted and/or the population becomes overcrowded.

Thus as N increases, we expect that r will decrease.

Moreover it can even become negative, meaning that the death rate exceeds the birth rate.

Graphically, we expect that $r(N)$ behaves like

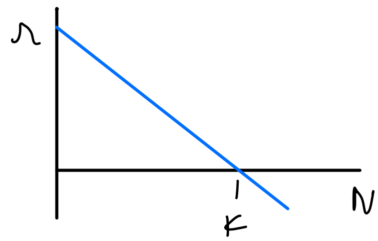


The point at which $r(N) = 0$ is a special population size which neither grows nor decays. It corresponds to $N = K$, where

$$K = \text{carrying capacity.}$$

In most cases we can't really know the shape of $r(N)$, but the notion of a carrying capacity remains reasonable.

So the simplest assumption would be to assume that $r(N)$ decreases linearly:



Then

$$r(N) = r\left(1 - \frac{N}{K}\right)$$

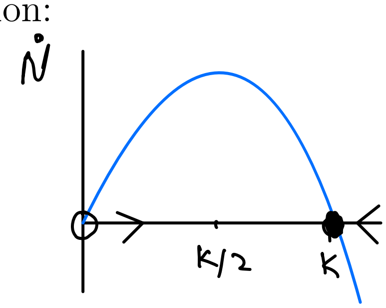
and our growth model now reads

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$

known as the *logistic equation*.*

The logistic equation can be solved exactly but in this course we are generally interested in qualitative analyses that lead to physical (and mathematical) insight.

Here, qualitative analysis begins by plotting $\mathrm{d}N/\mathrm{d}t$ vs. N, i.e., we plot the RHS of the logistic equation:



We consider only $N \geq 0$ since populations cannot be negative.

*The Belgian mathematician Verhulst published and named it in 1838, but he gave no reason for the name.

The fixed points occur where $\dot{N} = 0$. We find

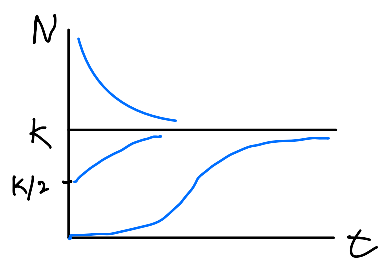
$$N_1^* = 0, \qquad N_2^* = K.$$

The slope of the curve or the directions of flow tell us that $N_1^* = 0$ is *unstable* $N_2^* = K$ is *stable*.

Indeed, we see that as long as we initiate growth with $N(0) > 0$, the population always evolves to the carrying capacity.

How it evolves depends on the initial condition:

- Small populations $N < K$ grow in an accelerating fashion, and then later slowly approach K. This is sometimes called S-shaped or *sigmoid*-shaped growth.
- Large populations $N > K$ or populations $K/2 < N < K$ continously decelerate as they approach K.



1.2 Linear stability analysis

We now address stability quantitatively. Recall that we are studying one-dimensional flows

$$\dot{x} = f(x).$$

Let

$$\eta = x - x^*$$

be a small perturbation away from a fixed point x^*.

Now note that

$$\begin{aligned}
\dot{\eta} &= \frac{\mathrm{d}}{\mathrm{d}t}(x - x^*) \\
&= \dot{x} \\
&= f(x) \\
&= f(x^* + \eta).
\end{aligned}$$

Expand $f(x^* + \eta)$:

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2).$$

Since x^* is a fixed point, $f(x^*) = 0$. Then

$$f(x^* + \eta) = \eta f'(x^*) + O(\eta^2).$$

Assuming that $f'(x^*) \neq 0$, the terms of $O(\eta^2)$ are negligible and we have

$$\dot{\eta} = \eta f'(x^*)$$

This simple linear equation shows us what we determined earlier from the slope of $f(x)$ in our graphs:

- $f'(x^*) < 0 \Rightarrow x$ decays exponentially $\Rightarrow x^*$ is *stable*.
- $f'(x^*) > 0 \Rightarrow x$ grows exponentially $\Rightarrow x^*$ is *unstable*.

But we now learn something else: the *characteristic timescale* of the exponential growth or decay is $1/|f'(x^*)|$.

What happens when $f'(x^*) = 0$? The higher-order terms then matter; we look at this later.

Example: Let's return to logistic growth. Here

$$f(N) = rN\left(1 - \frac{N}{K}\right)$$

with fixed points $N^* = 0$ and $N^* = K$. Since

$$f'(N) = r\left(1 - \frac{2N}{K}\right)$$

we find

$$f'(0) = r \quad \text{and} \quad f'(K) = -r.$$

As we found graphically, $N^* = 0$ is unstable and $N^* = K$ is stable.

Moreover, the timescale of growth or decay is $1/r$.

2 Bifurcations in one dimension

Reference: Strogatz, Chapter 3 [1].

Thus far, in our discussion of $\dot{x} = f(x)$, nothing terribly interesting happens: either $x \to x^*$ or $x \to \pm\infty$.

Now we consider instances in which $f(x)$ depends on a *control parameter* in a way that fixed points can be created or destroyed, or their stability can change.

Such qualitative changes are called *bifurcations.*

Examples: stick-slip motion as function of applied stress; the transition from conduction to convection as a fluid is heated from below.

Although the representation of bifurcations in one dimension may seem artificial or too simple, it turns out to be highly relevant in more general, multi-dimensional settings.

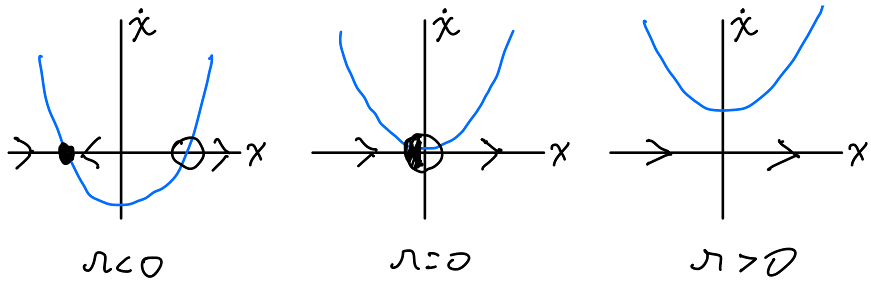
2.1 Saddle-node bifurcation

2.1.1 Normal form

Here we illustrate how fixed points can be created or destroyed. We write

$$\dot{x} = r + x^2$$

where r, the control parameter, can be positive, negative, or zero. The three cases are qualitatively different:



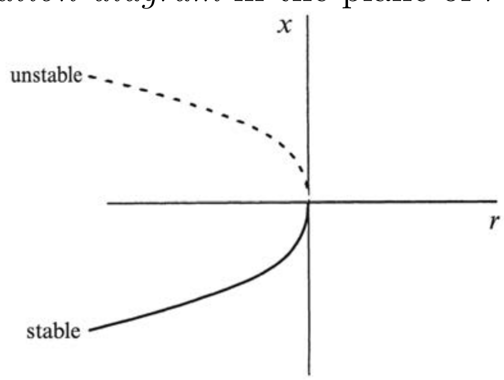
The bifurcation occurs at $r = 0$.

Fixed points exist only if $r < 0$. We then have

$$x^* = \pm\sqrt{-r}, \qquad r < 0$$

We can infer their stability graphically.

Now let's plot a *bifurcation diagram* in the plane of r and x.



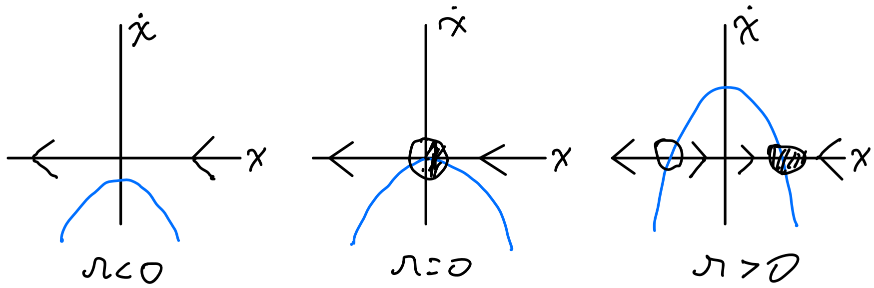
Strogatz [1], Fig. 3.1.4 See image credit on Page 19.

This type of bifurcation is called a *saddle-node bifurcation* (or a fold bifurcation). It occurs quite commonly. Let's consider a few other ways in which it can appear.

The first is trivial: write

$$\dot{x} = r - x^2$$

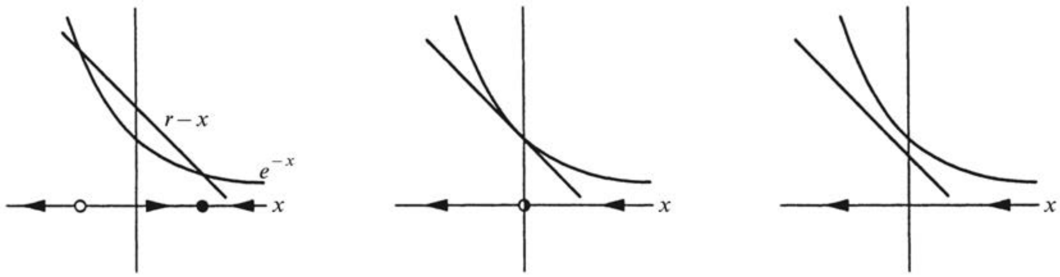
This is simply the reverse case: instead of two fixed points disappearing as r is increased past zero, the parabolas are concave down, two fixed points appear as r is increased past zero, and their stability is exchanged relative to the case above:



The second amounts to a tilt. Suppose that

$$\dot{x} = r - x - e^{-x}$$

Consider separate plots of $r - x$ and e^{-x}:



Strogatz [1], Fig. 3.1.6 See image credit on Page 19.

Fixed points occur at the intersections of the line and the curve, and their stability corresponds to the case we just considered.

The bifurcation occurs where the graphs of $r-x$ and e^{-x} intersect *tangentially*. This requires that

$$r - x = e^{-x}$$

and that their slopes are equal:

$$\frac{\mathrm{d}}{\mathrm{d}x}(r - x) = \frac{\mathrm{d}}{\mathrm{d}x}e^{-x}.$$

The second condition gives

$$-1 = -e^{-x} \quad \Rightarrow \quad x = 0$$

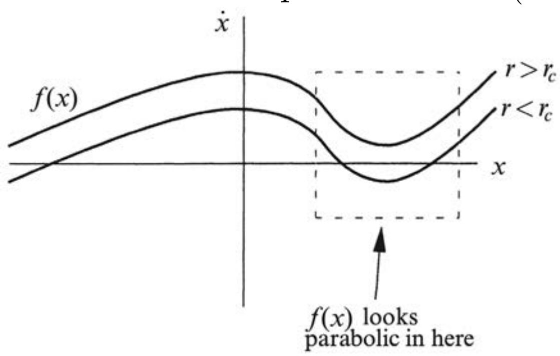
and the first gives $r = 1$; therefore the bifurcation occurs at $x = 0$ and $r_c = 1$.

We now show why the quadratic form is general (and is therefore called a *normal form*). Near the bifurcation, we write

$$
\begin{aligned}
\dot{x} &= r - x - e^{-x} \\
&= r - x - \left(1 - x + \frac{x^2}{2} + \dots\right) \\
&= r - 1 - \frac{x^2}{2} + \dots
\end{aligned}
$$

This has the same form as $\dot{x} = r - x^2$, and could agree exactly by rescaling r and x.

Essentially we are saying that $\dot{x}$ appears parabolic near $r = r_c$, where r_c is the critical value of r where the fixed points collide (or first appear).



Strogatz [1], Fig. 3.1.7 See image credit on Page 19.

This point can be made precise by Taylor expansion of $f(x, r)$ near $x = x^*$ for $r = r_c$:

$$
\begin{aligned}
\dot{x} &= f(x, r) \\
&= f(x^*, r_c) + (x - x^*) \left.\frac{\partial f}{\partial x}\right|_{x^*, r_c} + (r - r_c) \left.\frac{\partial f}{\partial r}\right|_{x^*, r_c} + \frac{1}{2}(x - x^*)^2 \left.\frac{\partial f}{\partial x^2}\right|_{x^*, r_c} + \dots
\end{aligned}
$$

where we have neglected terms of $O(x - x^*)^3$ and $O(r - r_c)^2$.

The first term vanishes because x^* is a fixed point.

The second term vanishes because the assumption of a saddle-node bifurcation requires that $f(x)$ be tangent to the x-axis at $r = r_c$. Then

$$\dot{x} = a(r - r_c) + b(x - x^*)^2$$

where a and b can be read off the Taylor expansion. Appropriate rescalings then bring us back to the normal form $\dot{x} = r + x^2$.

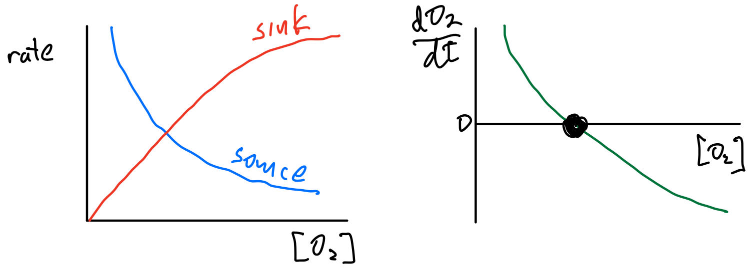
2.1.2 Example: oxygenation of the atmosphere

Many problems of dynamics may be expressed as

$$\dot{x} = \text{inputs} - \text{outputs}$$

As an example, let's ask how the atmosphere became oxygenated.

Today's level of O_2 is stable. Conceptually that means there are sources and sinks that balance in a stable way [2]:

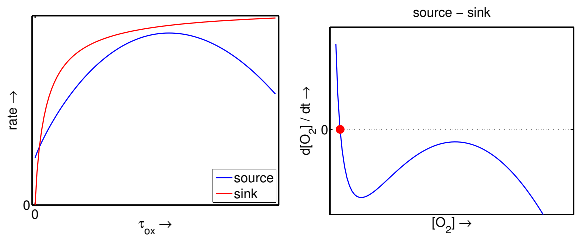


There was essentially no O_2 in the atmosphere until about 2.5 billion years ago, when oxygen-producing photosynthesis developed.

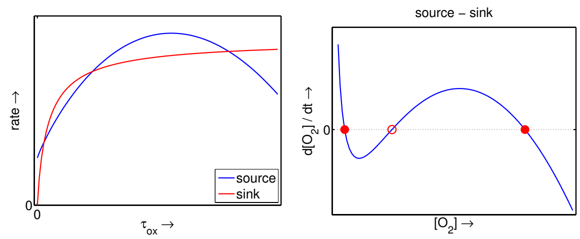
But the first aerobic organisms presumably consumed it voraciously. And we know from various observations that O_2 levels remains low for at least hundreds of millions of years.

How, then, could it have increased if the stability argument just given is correct?

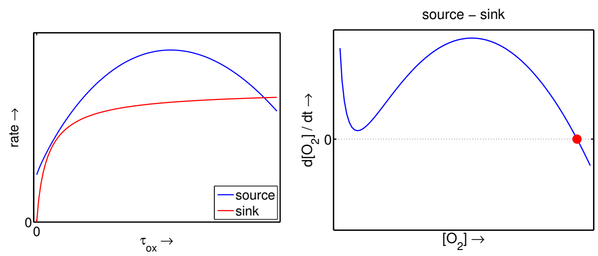
One idea is that the source function had a non-monotonic dependence on O_2 levels, and the sink function became weaker with time. A first we would have a situation like this:



Then, as the sink function weakened, two new fixed points would appear (via a first saddle-node bifurcation):



Further weakening of the sink then causes the low-O_2 fixed point to collide with the unstable fixed point via a second saddle-node bifurcation, leaving only the high-O_2 state:



For further details and evidence that such a picture may be correct, see Ref. [3].

2.2 Transcritical bifurcation

In some problems a fixed point must always exist. For example, in the logistic equation, the zero-population fixed point must always be present.

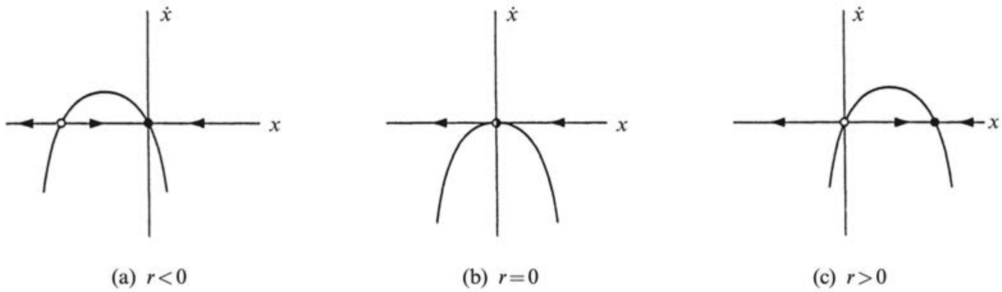
However the stability of the permanent fixed point may change as a parameter is varied.

The normal form of the transcritical bifurcation accounts for this possibility. Its normal form reads

$$\dot{x} = rx - x^2$$

This looks like the logistic equation, but now x and r are no longer constrained to be positive.

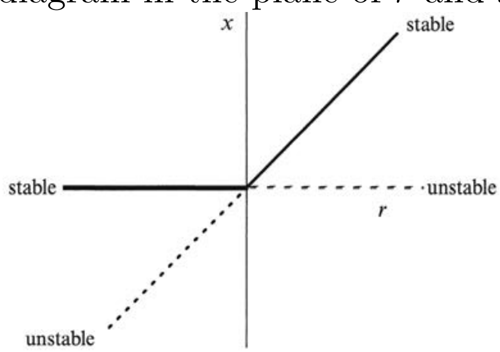
Graphically, we have



Strogatz [1], Fig. 3.2.1 See image credit on Page 19.

We see that the fixed point $x^* = 0$ always occurs, but as r increases past zero the unstable fixed point $x^* = r$ for $r < 0$ becomes stable when $r > 0$.

Here's the bifurcation diagram in the plane of r and x:



Strogatz [1], Fig. 3.2.2 See image credit on Page 19.

2.3 Pitchfork bifurcation

Many physical problems exhibit symmetry, such that our one dimensional dynamics are invariant under $x \to -x$. Examples:

- *Buckling beam.* Put a weight on a beam confined between two rigid walls. If the weight is sufficiently heavy, the beam will buckle to either the left or right.
- *Thermal convection.* Heat a fluid from below. Eventually, hot fluid rises and cold fluid sinks, forming convective "rolls." In two dimensions, the motion may be either clockwise or counter-clockwise.

Such symmetry leads to a *pitchfork bifurcation.* There are two types: supercritical and subcritical.

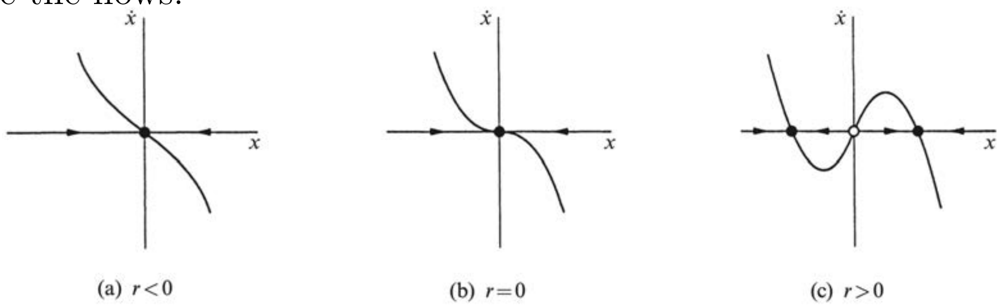
2.3.1 Supercritical pitchfork bifurcation

The normal form is

$$\dot{x} = rx - x^3$$

Note that the equation is invariant for $x \to -x$.

Here are the flows:



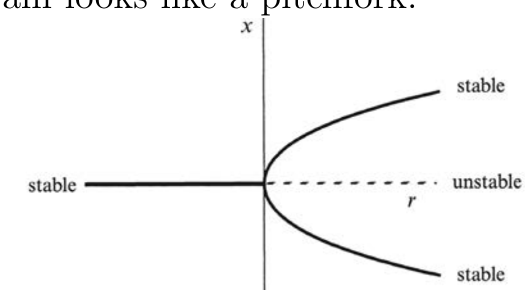
Strogatz [1], Fig. 3.4.1 See image credit on Page 19.

When $r < 0$, there is only one fixed point, $x^* = 0$.

When $r = 0$, the origin is still stable, but only weakly because the linear term has vanished. Thus there is no longer an exponential decay towards $x^* = 0$. This is called *critical slowing down*.

When $r > 0$, the origin becomes unstable and two new fixed points occur, at $x^* = \pm\sqrt{r}$. Both are stable.

The bifurcation diagram looks like a pitchfork:



Strogatz [1], Fig. 3.4.2 See image credit on Page 19.

Note that there is now bistablity: depending on the initial condition, we either end up at the positive or negative fixed point. Thus the initial condition "breaks" the symmetry.

Another way of viewing the problem is to derive the dynamics $\dot{x}$ from a potential $V(x)$. We define the potential by

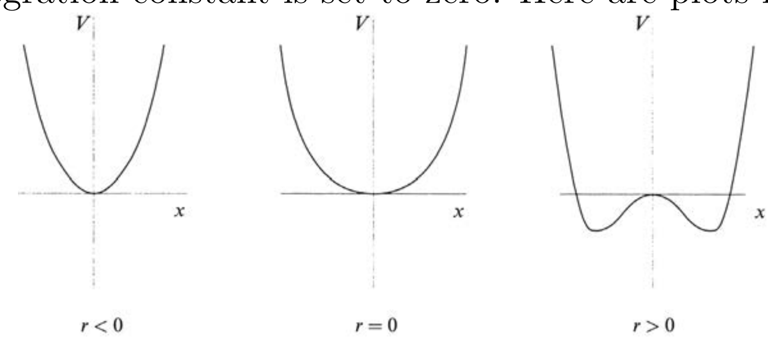
$$\dot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

so that x tends to go in the direction of smaller V (i.e., downhill), without inertia, at a rate proportional to the slope of the potential.

Integrating the right-hand side, we obtain

$$V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4,$$

where the integration constant is set to zero. Here are plots for different r:



Strogatz [1], Fig. 3.4.5 See image credit on Page 19.

This sort of potential occurs in studies of phase transitions where either of two states are possible, such as spontaneous magnetization or phase separation. In this case the parameter r is typically the temperature, and the critical temperature corresponds to $r = 0$.

We can also now better appreciate the notion of critical slowing down: at $r = 0$ the potential is nearly flat near the origin, so fluctuations relax very slowly.

2.3.2 Subcritical pitchfork bifurcation

In the supercritical case, the cubic term is stabilizing: its sign is opposite that of the instability caused by the linear term.

To see what this means, consider x very small; then the nonlinear term is negligible so that, for $r > 0$,

$$\dot{x} \sim rx \quad \Rightarrow \quad x \sim e^{rt}.$$

But eventually the cubic term becomes significant. If we write the supercritical normal form as

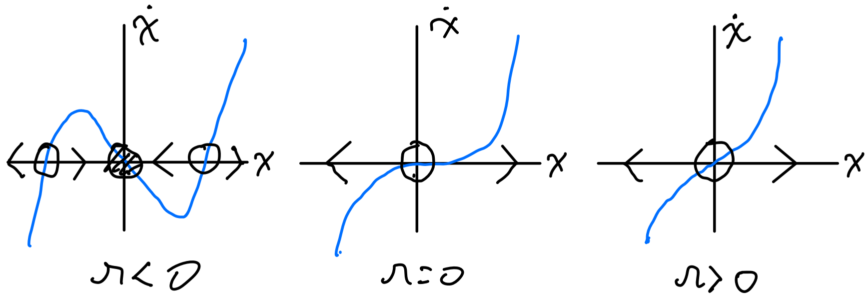
$$\dot{x} = rx\left(1 - \frac{x^2}{r}\right),$$

we see that "saturation" of the instability occurs when $x = \pm\sqrt{r}$, i.e., at a fixed point.

However, nothing prevents the lowest-order nonlinear term from also being destabilizing. Thus we could have

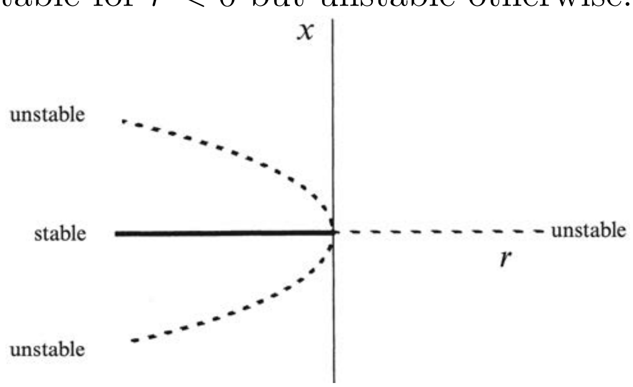
$$\dot{x} = rx + x^3.$$

The flows are now flipped compared to previously:



The non-zero fixed points $x^* = \pm\sqrt{-r}$ are now unstable, and they exist only for $r < 0$. (Thus we see some reason for the name "subcritical.")

And the origin is stable for $r < 0$ but unstable otherwise:



Strogatz [1], Fig. 3.4.6 See image credit on Page 19.

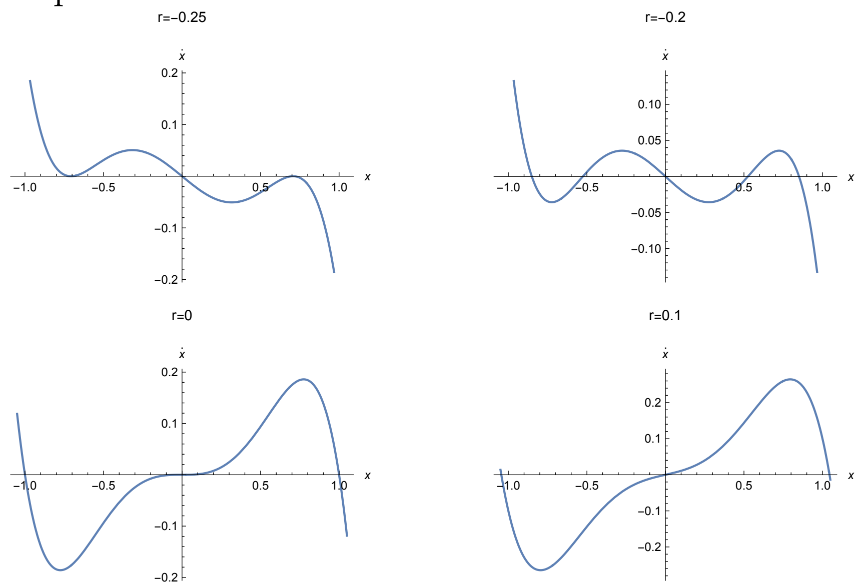
Of course, now the cubic term drives trajectories out to $\pm\infty$, so we still need a stabilizing term.

The lowest-order nonlinear term that still maintains symmetry is then x^5, leading to the normal form

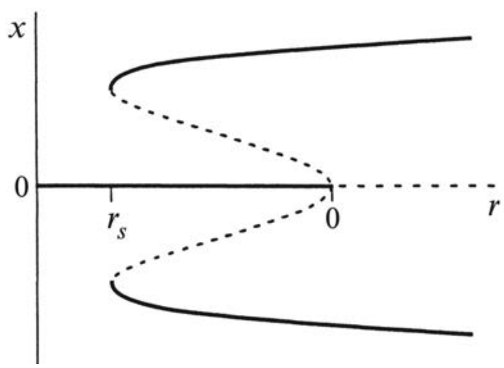
$$\dot{x} = rx + x^3 - x^5.$$

Although things now become somewhat complicated algebraically, the effect of the damping term must create a new set of *stable* fixed points above the dashed lines.

Here are some plots for different values of r:



The bifurcation diagram then looks like this:



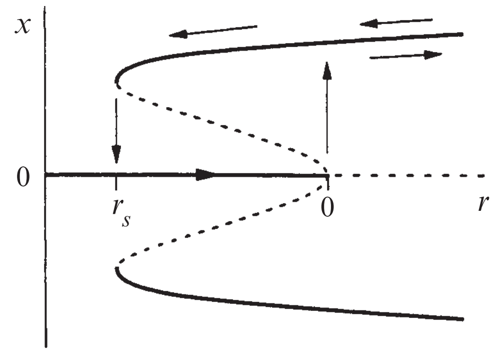
Strogatz [1], Fig. 3.4.7 See image credit on Page 19.

Here $r_s = -0.25$. Otherwise there is plenty to say:

- When $r_s < r < 0$, both the origin and the new large-amplitude fixed points are stable. The initial condition determines the final outcome. Thus the origin is stable to small perturbations, but not to large perturbations.

- *Jumps.* When r is increased to $r = 0$, the system jumps to a large-amplitude fixed point.

- *Hysteresis.* If r is then decreased, the system *stays* on the large amplitude branch until it reaches r_s, and then it jumps back to the origin. The dynamics are not reversible.



Strogatz [1], Fig. 3.4.8 See image credit on Page 19.

- As r is increased past r_s from below, both an unstable and a stable fixed point appear, below and above the origin. Our earlier discussion of such spontaneous appearance of stable and unstable fixed points implies that this occurs by a saddle-node bifurcation, as one can see in the plot of $\dot{x}$ vs. x for $r = -0.25$.

A major difference between the supercritical and subcritical pitchfork bifurcation is that the former leads to continuous changes $x^* \sim \sqrt{r}$ as r is increased past zero, whereas the latter results in a finite jump.

Because the jump can be quite large, subcritical bifurcations are sometimes called hard or dangerous, while supercritical bifurcations are called soft or safe.

And in the analogy with phase transitions, supercritical bifurcations are related to continuous or second-order phase transitions (e.g., the ferromagnetic transition), while subcritical bifurcations are discontinuous or first-order (e.g., the freezing of water into ice).

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).

2. Holland, H. D. *The Chemistry of the Atmosphere and Oceans* (John Wiley & Sons, New York, 1978).
3. Shang, H., Rothman, D. H. & Fournier, G. P. Oxidative metabolisms catalyzed Earth's oxygenation. *Nature communications* **13,** 1–9 (2022).

Image Credit

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