Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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1 Bifurcations in two dimensions

Reference: Strogatz, Chapter 8 [1].

We now extend our earlier discussion of bifurcations in 1-D to 2-D.

Unlike 1-D, where trajectories either stop or go to infinity, now we shall meet the class of bifurcations that create limit cycles.

First let's generalize our earlier results.

1.1 Saddle-node bifurcation

Recall that saddle-node bifurcations create or destroy fixed points. The 2-D prototype is

$$\begin{aligned} \dot{x} &= \mu - x^2 \\ \dot{y} &= -y \end{aligned}$$

All that we've done is add the equation for $\dot{y}$, which sends trajectories toward the x-axis.

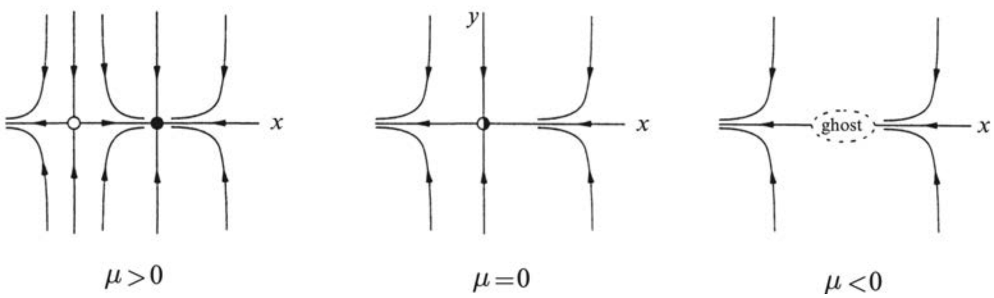
When $\mu < 0$, there is a stable fixed point at

$$(x^*, y^*) = (\sqrt{\mu}, 0)$$

and an unstable fixed point at

$$(x^*, y^*) = (-\sqrt{\mu}, 0)$$

The phase portraits show their collision at $\mu = 0$:



Strogatz [1], Fig. 8.1.1 See image credit on Page 12.

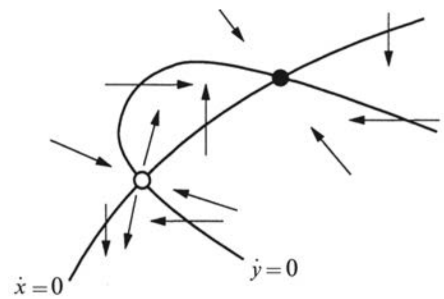
The ghost is a kind of bottleneck: even after the fixed points disappear, the fact that they had been present causes the flow to be slow.

To see how slow it is, we compute the time taken to move along the x-axis in the one-dimensional case with $\mu < 0$:

$$\begin{aligned} T &= \int_{-\infty}^{\infty} \frac{1}{\mathrm{d}x/\mathrm{d}t} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu - x^2} \mathrm{d}x \\ &= \frac{\pi}{\sqrt{-\mu}} \end{aligned}$$

which diverges as $\mu \to 0$ from below. Exercise 4.3.1 in Strogatz shows how to evaluate the integral.

While our prototypical example may appear special, it is really just a simple example of a more general 2D system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ in which the nullclines for $\dot{x} = 0$ and $\dot{y} = 0$ intersect like this:



Strogatz [1], Fig. 8.1.2 See image credit on Page 12.

Then as a control parameter is varied, the nullclines begin to separate, the two fixed points must collide, and the system would behave as above.

1.2 Transcritical and pitchfork bifurcations

In analogy with the saddle-node case (and our earlier work in 1-D), the prototypical cases are

$$
\begin{array}{lll}
\dot{x} = \mu x - x^2, & \dot{y} = -y & \text{transcritical} \\
\dot{x} = \mu x - x^3, & \dot{y} = -y & \text{supercritical pitchfork} \\
\dot{x} = \mu x + x^3, & \dot{y} = -y & \text{subcritical pitchfork}
\end{array}
$$

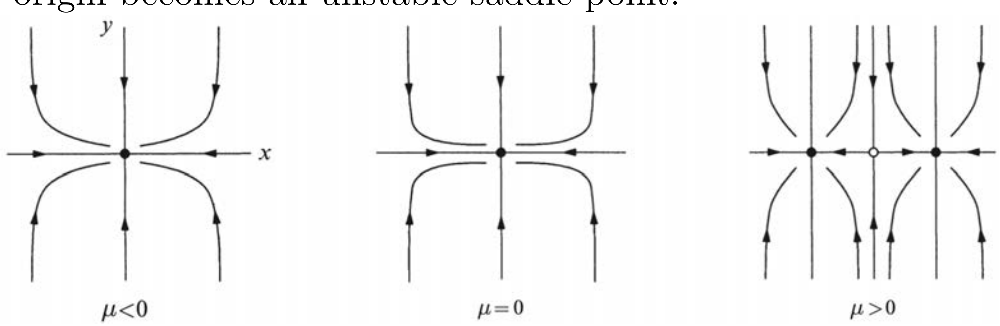
The pitchfork bifurcations will be most useful for us. Let's look at the supercritical case.

For $\mu \leq 0$, there is a stable fixed point at the origin.

For $\mu > 0$, two new stable fixed points appear, at

$$(x^*, y^*) = (\pm\sqrt{\mu}, 0)$$

and the origin becomes an unstable saddle point:



Strogatz [1], Fig. 8.1.6 See image credit on Page 12.

1.3 Hopf bifurcations

Thus far our examples of bifurcations have all been expressed in terms of collisions of fixed points on a line, even though the line may exist as a curve in the plane. We now describe a bifurcation that can only exist in a phase space of two or more dimensions.

In general, a stable fixed point may become unstable only if the real part of one of the eigenvalues λ of the Jacobian becomes greater than zero as the control parameter μ varies.

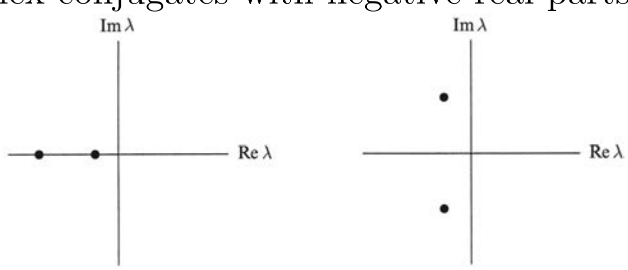
Recall that the eigenvalues λ solve

$$\det J - \lambda I = 0,$$

where J is the Jacobian and I is the identity matrix.

In 2-D, the resulting quadratic equation either has two real roots or two complex conjugate roots.

So when a fixed point is stable, either both roots are real and negative, or the roots are complex conjugates with negative real parts:



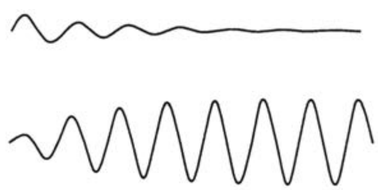
Strogatz [1], Fig. 8.2.1 See image credit on Page 12.

In the saddle-node, transcritical, and pitchfork bifurcations, one of the purely real roots passes through $\lambda = 0$ when the fixed point becomes unstable.

A *Hopf bifurcation* occurs in the case in which the complex conjugate roots cross the imaginary axis. As in pitchfork bifurcations, there are two cases: supercritical and subcritical.

1.3.1 Supercritical Hopf bifurcation

The complex eigenvalues produce oscillatory solutions. One possibility is that oscillations are damped for $\mu < \mu_c$ and growing for $\mu > \mu_c$:



Strogatz [1], Fig. 8.2.2 See image credit on Page 12.

This situation corresponds to a *supercritical Hopf bifurcation*.

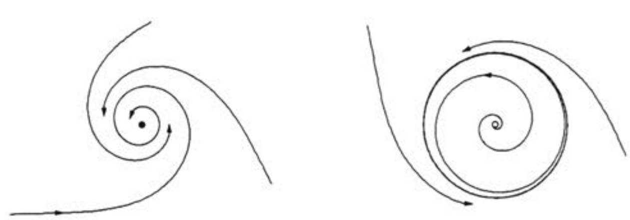
We can write a simple model of the supercritical Hopf bifurcation in terms of the radius r and angle θ in the 2-D phase space:

$$\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + b r^2\end{aligned}$$

Here μ controls the stability, ω gives the frequency of oscillations when r is infinitesimal, and b determines the dependency of the frequency on the amplitude of large oscillations.

Note the similarity of the r-equation with the supercritical pitchfork bifurcation; the θ equation provides a kind of driving.

When $\mu \leq 0$, the origin $r = 0$ is a stable spiral. When $\mu > 0$, the origin is an unstable spiral:



Strogatz [1], Fig. 8.2.3 See image credit on Page 12.

Note that for $\mu > 0$ there is a stable limit cycle of size $r = \sqrt{\mu}$.

When $\mu = 0$ the origin is still stable, but the amplitude of oscillations decays slower than exponentially (since $\dot{r} = -r^2$); this is another case of critical slowing down.

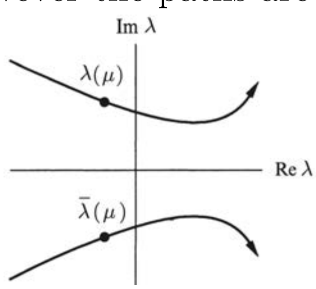
The eigenvalues at the origin are (unsurprisingly, but see the calculation in Strogatz)

$$\lambda = \mu \pm i\omega$$

We can extract two important characteristics of supercritical Hopf bifurcations from this prototypical form:

- The size of the limit cycle grows like $\sqrt{\mu}$ for μ near μ_c.
- The frequency of the limit cycle near μ_c is approximately $\omega = \text{Im}\lambda$, evaluated near at μ_c. The period is $2\pi/\omega$.

In our example, the eigenvalues cross the imaginary axis as a straight line parallel to the real axis. However the paths are usually curved:



Strogatz [1], Fig. 8.2.4 See image credit on Page 12.

Moreover the limit cycle is usually elliptical and not circular near μ_c.

1.3.2 Subcritical Hopf bifurcation

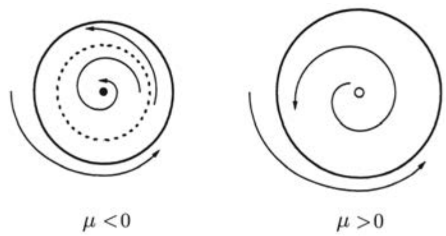
As with pitchfork bifurcations, there is also a subcritical variety of Hopf bifurcations in which the stability of the fixed point and the limit cycle is reversed. It has some very interesting properties.

Here's the prototype:

$$\begin{aligned}\dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2\end{aligned}$$

Now the cubic term is destabilizing—it helps drive trajectories away from the origin.

The phase portraits look like this:



Strogatz [1], Fig. 8.2.5 See image credit on Page 12.

When $\mu < 0$ there are two attractors: a stable limit cycle and a stable fixed point. These two attractors are separated by an *unstable* limit cycle.

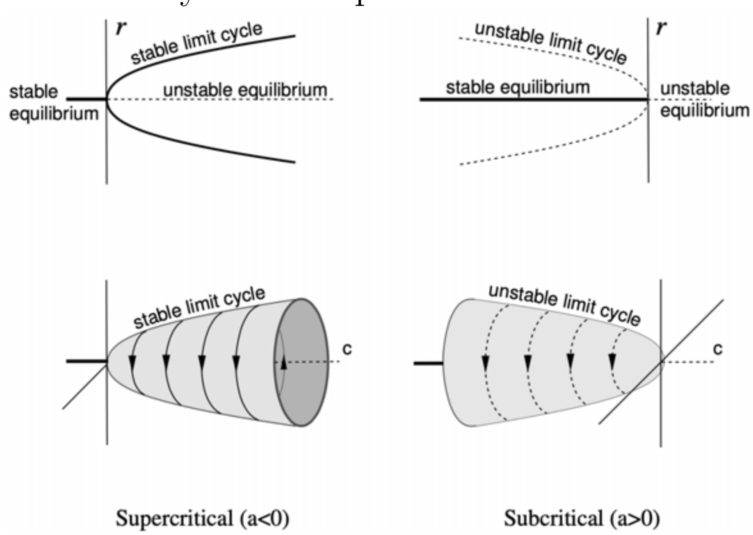
As $\mu \to 0$ from below, the unstable limit cycle shrinks to the origin.

The *subcritical Hopf bifurcation* occurs at $\mu = 0$. Here the origin becomes *unstable* and the large amplitude limit cycle is the only attractor.

Consequently any solution near the origin is forced to grow immediately to a large amplitude oscillation.

This strongly contrasts the supercritical case, in which the amplitude of the limit cycle grows smoothly from zero, like $\mu^{1/2}$.

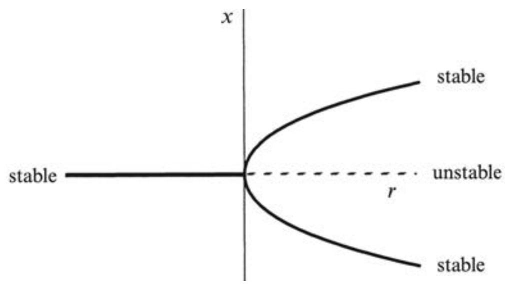
Here's a pictorial summary of the supercritical and subcritical cases:



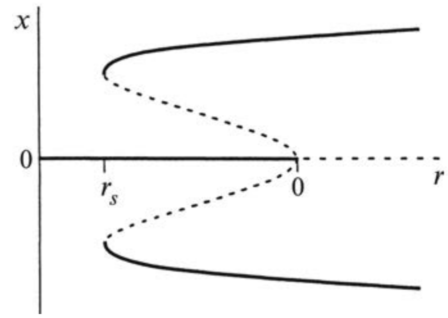
Izhikevich [2], Fig. 6.9

Note that pictures of the subcritical case omit the large amplitude stable limit cycle.

Note the similarity to the supercritical and subcritical pitchfork bifurcations in one dimension we saw previously:



Strogatz [1], Fig. 3.4.2



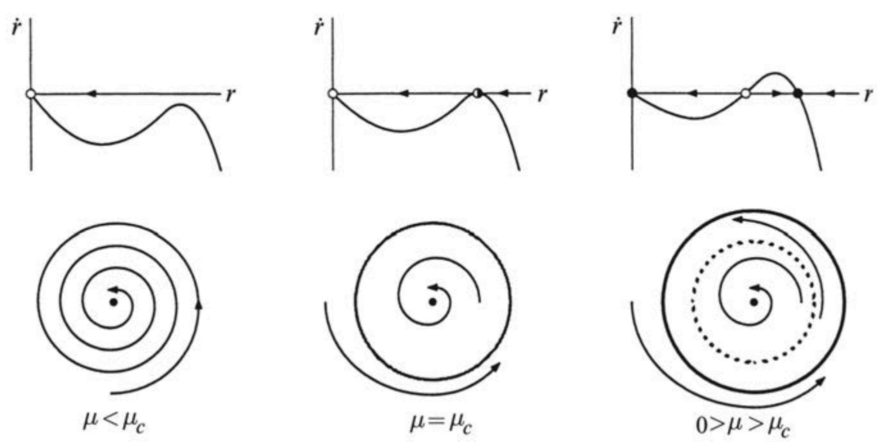
Strogatz [1], Fig. 3.4.7 See image credit on Page 12.

As in the subcritical pitchfork bifurcation, there is also the possibility of hysteresis: once trajectories are on the large amplitude limit cycle for $\mu > 0$, they can't be turned off by returning μ to zero.

It turns out the that the large-amplitude limit cycle disappears at $\mu_c = -1/4$. How does that happen?

1.4 Saddle-node bifurcation of cycles

We can answer that question by considering only the equation for $\dot{r}$. Plots of $\dot{r}$ for $\mu < \mu_c$, $\mu = \mu_c$, and $\mu_c < \mu < 0$ look like this:



Strogatz [1], Fig. 8.4.1 See image credit on Page 12.

Here the inclusion of the phase portraits makes clear that a saddle-node bifurcation occurs at $\mu = \mu_c$.

Rather than just being a collision of an unstable and stable fixed point, here we have a *saddle node bifurcation of cycles*: the unstable limit cycle collides with the stable limit cycle, leaving only the stable fixed point.

In fact the origin remains stable all the time, while the large amplitude limit cycle is created, seemingly from nothing, as μ increases past μ_c.

1.5 Spiking near limit cycles: excitability

Recall that the van der Pol equation

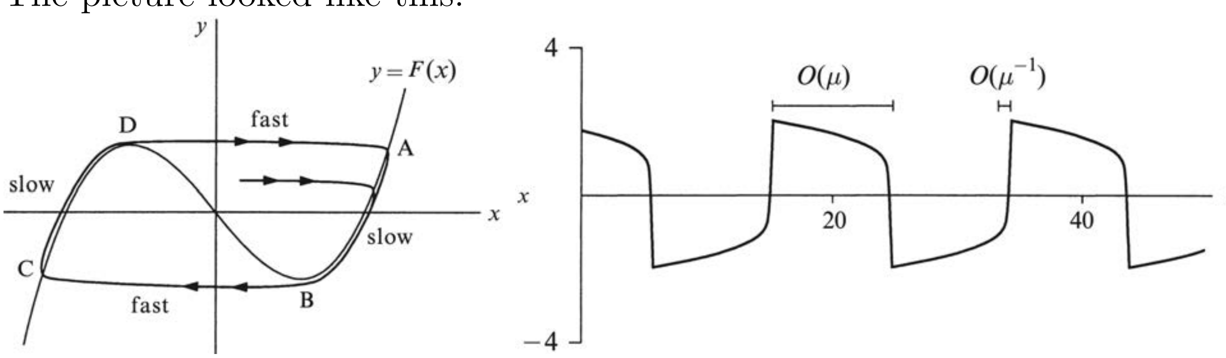
$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

may be rewritten as

$$\begin{aligned} \dot{x} &= y + x - \frac{x^3}{3} \\ \dot{y} &= -\frac{1}{\mu}x \end{aligned}$$

We showed earlier, via a physical argument, that the origin or rest position is always unstable, and that a limit cycle is instead stable.

We also showed that when $\mu \gg 1$, the system exhibits relaxation oscillations. The picture looked like this:



Strogatz [1], Figs. 7.5.1-2 See image credit on Page 12.

Here the $\dot{x} = 0$ nullcline is the cubic. The $\dot{y} = 0$ nullcline is the y-axis (i.e., $x = 0$). Their intersection is at the unstable fixed point, the origin.

We can make the fixed point stable, however, by tilting the $\dot{y} = 0$ nullcline. Along with a few other modifications of only cosmetic consequence, the modified system reads

$$\begin{aligned}\dot{x} &= -y + x - x^3/3 + z \\ \dot{y} &= \frac{1}{\mu}(x + a - by)\end{aligned}$$

Comparing with the van der Pol equation, one obvious but inconsequential difference is the change $y \to -y$, which flips the cubic nullcline.

The $\dot{y} = 0$ nullcline is now the line

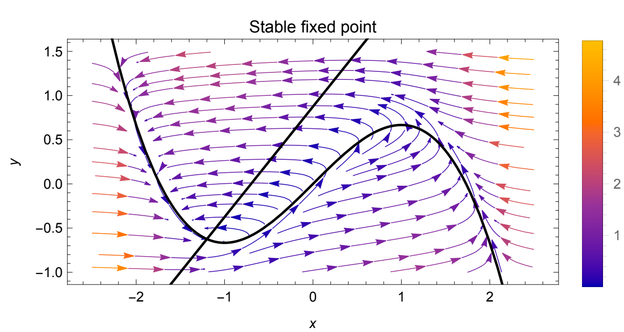
$$y = \frac{x + a}{b}$$

which not only provides a tilt of slope $1/b$, but also adds a constant a/b. We have also added a constant term z to the first equation.

This system is known as the Fitzhugh-Nagumo model [2–4]. It was introduced as a simple model of an action potential, which describes the firing of pulses or spikes of voltage, as is commonly observed in the behavior of neurons. The variable x is then loosely related to the membrane potential and y is called a recovery variable.

The Fitzhugh-Nagumo model was originally introduced as a simple way to explain the behavior of the more detailed *Hodgkin-Huxley model* of the squid giant axon [2].

Here we are interested in the model when the nullclines intersect at only one point and that point is stable. Here's how the flow appears in the phase plane:



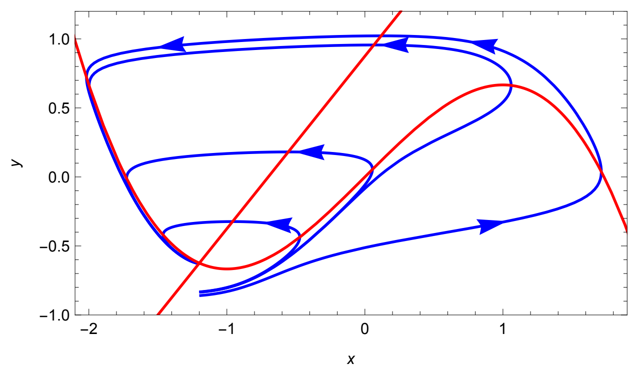
Here the parameters $a = 0.7$, $b = 0.8$, $\mu = 10$, and $z = 0$.

If we were to increase z, the effect would be like stimulating the system with a current. Eventually the system passes through a Hopf bifurcation, and the system would exhibit a limit cycle very similar to the van der Pol limit cycle.

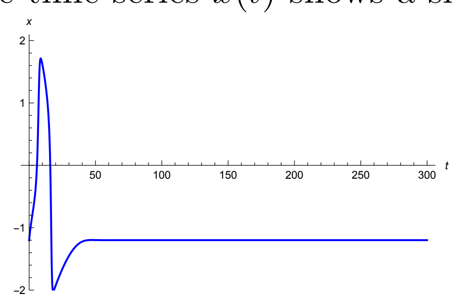
But note above that the flow in phase space when the fixed point is stable looks very similar to what one would expect for a limit cycle.

Consequently, *near* the Hopf bifurcation, perturbations of the fixed point above a threshold make the system behave as if the limit cycle existed, but only for one cycle.

Here are four perturbations of the fixed point, two below the threshold and two above the threshold:



Above the threshold, the time series $x(t)$ shows a single pulse or spike:



Systems that exhibit such spiking behavior are called *excitable*. Models of excitable systems have played a large role in neurophysiology. They may also apply to other systems, including the climate and carbon cycle [5].

In the case we have considered here, for practical purposes the threshold for an excitation is sharp, but mathematically it is smooth (as can be seen in the phase space trajectories above), and is thus called a *quasithreshold*.

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).
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3. FitzHugh, R. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal* **1,** 445–466 (1961).
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5. Rothman, D. H. Characteristic disruptions of an excitable carbon cycle. *Proceedings of the National Academy of Sciences* **116,** 14813–14822 (2019).

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