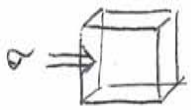
12.002 Physics and Chemistry of the Earth and Terrestrial Planets
Fall 2008

12.002 fall 2008 p-set 5 Solutions

① $Ra = \dfrac{d^3 \alpha \rho g \Delta T}{\mu \kappa}$

$$[Ra] = \frac{[\cancel{m^3}][\frac{1}{\cancel{°C}}][\frac{\cancel{kg}}{\cancel{m^3}}][\frac{\cancel{m}}{\cancel{s^2}}][\cancel{°C}]}{[\frac{\cancel{kg}}{\cancel{ms}}][\frac{\cancel{m^2}}{\cancel{s}}]} = 1 \quad \text{dimensionless!}$$

②



$\sigma = 10^{-3} MPa = 10^3 Pa$

$\sigma = \dfrac{F}{Area}$ Area of the side of the cube : $A = 10^{12} m^2$

$F = \sigma \cdot A = 10^3 Pa \cdot 10^{12} m^2 = 10^{15} N$

$F = ma \Rightarrow a = \dfrac{F}{m}$ $m = V \cdot \rho$ $\rho = 4000 \frac{kg}{m^3}$

$V = 10^{18} m^3$

$m = 4 \cdot 10^{21} kg$

$$a = \frac{10^{15} N}{4 \cdot 10^{21} kg} = 0.25 \cdot 10^{-6} \frac{m}{s^2}$$

$$a = \frac{\Delta v}{\Delta t}^{*} \Rightarrow \Delta v = \underset{\text{in year}}{\Delta t}\, a = 60 \cdot 60 \cdot 24 \cdot 365 s \cdot 0.25 \cdot 10^{-6} \frac{m}{s^2} = \boxed{7.884 \ m/s} \approx 2.49 \cdot 10^{11} \ mm/yr$$

* assume initial velocity 0

very fast when compared to 100 mm/yr

As we don't observe such speeds in the mantle, the net forces (stresses) acting on the material must be virtually 0.

③

a) $Q_{cond} = \dfrac{K \Delta T}{d}$

$Q_{conv} = \beta \dfrac{d^2 \rho^2 \Delta T^2 g \alpha C_p}{\mu}$

$K = \kappa \rho C_p$

$$\frac{Q_{conv}}{Q_{cond}} = \beta \cdot \frac{d^2 \rho^2 \Delta T^{\cancel{2}} g \alpha C_p}{\mu} \cdot \frac{d}{K \cancel{\Delta T}} =$$

$$= \beta \frac{d^3 \rho^{\cancel{2}} \Delta T g \alpha \cancel{C_p}}{\mu} \cdot \frac{1}{\kappa \cancel{\rho} \cancel{C_p}}$$

$$= \beta \frac{d^3 \alpha \rho g \Delta T}{\mu \kappa} = \underline{\underline{\beta \cdot Ra}}$$

dimensionless

b) The total heat flow is a sum of Q_{cond} and Q_{conv}. So if a layers have different ratios of $\dfrac{Q_{conv}}{Q_{cond}}$, they have different Ra even though $Q_{conv} + Q_{cond}$ is the same.

(4)

Approach 1. Calculate R from the steady state equation and check the Ra if the mantle is converting

Steady State: $0 = -\beta \frac{d\rho\Delta T^2 g d}{\mu} + \frac{Q_r R}{\rho c_p d}$ $\quad g = g_e \frac{R}{R_e} \quad d = \frac{R}{2}$

$$0 = -\beta \frac{R^2 \rho \Delta T^2 g_e \alpha}{2 R_e \mu} + \frac{2 Q_r}{\rho c_p}$$

$$R = \sqrt{\frac{4 Q_r R_e \mu}{\rho^2 c_p \beta \Delta T^2 g_e \alpha}} \rightarrow \text{plug in to get } R = \boxed{5634 \text{ km}}$$

$$Ra = \left(\frac{R}{2}\right)^3 \alpha \rho g_e \frac{R}{R_e} \Delta T \frac{1}{\mu \kappa} \rightarrow \text{plug in to get } Ra = \boxed{15747}$$

bigger then 1000 so it's converting

Approach 2: Put Ra into steady state eqn and determine minimum R for which mantle is converting:

$$0 = -\beta \frac{d\rho\Delta T^2 g \alpha}{\mu} + \frac{Q_r R}{\rho c_p d} \qquad Ra = \frac{d^3 \alpha \rho g \Delta T}{\mu \kappa}$$

$$0 = -\beta Ra \frac{\kappa \Delta T}{d^2} + \frac{Q_r R}{\rho c_p d} \Rightarrow R = \sqrt{\frac{2\beta Ra \kappa \Delta T \rho c_p}{Q_r}} \rightarrow \text{plug in to get } R = \boxed{449 \text{ km}} \text{ for } Ra = 1000$$

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% P4____________________________________________________

Qr = 10^-8;                     % W/m^3
ge = 10;                        % m/s^2
k = 10^-6;                      % m^2/s
rho = 4000;                     % kg/m^3
Cp = 1260;                      % J/kgK
K = 5;                          % W/mK
alpha = 10^-5;                  % 1/C
delta_T = 20;                   % C
mu = 10^21;                     % Pas
beta = 0.01;
Re = 6400000;                   % m

P4_R1 = sqrt(4*Qr*Re*mu/(rho^2*Cp*beta*delta_T^2*ge*alpha))

d = P4_R1/2;

P4_Ra1 = d.^3*alpha*rho*ge.*P4_R1/Re*delta_T/(mu*k)

Ra = [200 1000];

P4_R2 = sqrt(2*beta*Ra*k*delta_T*rho*Cp/Qr)

ANS_____________________________________________________

P4_R1 = 5.6344e+06

P4_Ra1 = 1.5747e+05

P4_R2 =
   1.0e+05 *
    2.0080    4.4900
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(5)

$$\underbrace{\ln\left(\frac{Q_r R \mu_0}{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha}\right) - \ln\left[1 - \exp\left(\frac{-t \gamma Q_r}{\rho C_p}\right)\right]}_{LHS} = \gamma (T - T_0)$$

$$T = \frac{LHS}{\gamma} + T_0$$

$$Q_{conv} = \frac{\beta \left(\frac{R}{2}\right)^2 \rho^2 \Delta T^2 g \alpha C_p}{\mu_0 e^{\gamma (T_0 - T)}}$$

plug the numbers in matlab (see below)

a) $\boxed{T = 1369\ ^\circ C}$ $\boxed{Q_{conv} = 0.064\ W/m^2}$

b) $\boxed{T = 1316\ ^\circ C}$ $\boxed{Q_{conv} = 0.0045\ W/m^2}$

c) Big change in Q_r (radioactive decay heat production) causes the temperature to change only slightly. Convection is the buffer that keeps the T at the same ~~[illegible]~~ order of magnitude: when Q_r increases, the convection increases so that there is more heat loss due to convection and the temperature is thus ~~decreased~~ maintained.

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% P5______________________________________________________

gamma = 0.05;
mu0 = 10^22;                    % Pas
T0 = 1300;                      % C
delta_T = 10;                   % C
t = 4.5*10^9*60*60*24*365;      % s

d=Re/2;

LHS = log(Qr*Re*mu0/(beta*d^2*rho^2*Cp*delta_T^2*ge*alpha)) - log(1 -
exp(-t*gamma*Qr/(rho*Cp)));

P5_T_a = LHS/gamma + T0

P5_q_conv_a = beta*d^2*rho^2*delta_T^2*ge*alpha*Cp/(mu0*exp(gamma*(T0-
P5_T_a)))

Qr = 10^-12;

LHS = log(Qr*Re*mu0/(beta*d^2*rho^2*Cp*delta_T^2*ge*alpha)) - log(1 -
exp(-t*gamma*Qr/(rho*Cp)));

P5_T_b = LHS/gamma + T0

P5_q_conv_b = beta*d^2*rho^2*delta_T^2*ge*alpha*Cp/(mu0*exp(gamma*(T0-
P5_T_b)))

ANS______________________________________________________________________
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a)

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P5_T_a = 1.3687e+03

P5_q_conv_a = 0.0640
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b)

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P5_T_b = 1.3158e+03

P5_q_conv_b = 0.0045
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